SOLUTIONS TO ADDITIONAL EXERCISES FOR II.1 AND II.2

Here are the solutions to the additional exercises in `betsepexercises.pdf`.

B1. Let $\mathbf{y}$ and $\mathbf{z}$ be distinct points of L; we claim that $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ are not collinear. If there were some line M containing them, then we would have $M = L$ since both lines contain the last two points; however, we know that $\mathbf{x} \notin L$, so this is impossible.

To show the existence of a plane containing L and $\mathbf{x}$, let P be the unique plane containing $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$. Since $\mathbf{y}$ and $\mathbf{z}$ are in P, the axioms imply that the line joining them, which is L, must be contained in P. To see that there is only one plane containing L and $\mathbf{x}$, notice that a plane Q which contains both of these will automatically contain $\mathbf{y}$ and $\mathbf{z}$. Since there is only one plane containing $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$, it follows that Q must be identical to P.■

B2. By the previous exercises, there is a unique plane P containing X and K. Let A, B, C be the points where K meets the lines L, M, N respectively. Then we have the following:

(1) Since X and A lie on P, the line $XA = L$ is contained in P.

(2) Since X and B lie on P, the line $XB = M$ is contained in P.

(3) Since X and C lie on P, the line $XC = N$ is contained in P.

Therefore P contains each of the lines K, L, M, N.■

B3. Since $A * B * C$ and $X * Y * Z$ are assumed, the conditions on the distances imply that

$$d(A,B) \;=\; d(A,C) - d(B,C) \;=\; d(X,Z) - d(Y,Z) \;=\; d(X,Y)$$

which is what we wanted to prove.■

B4. If $X \in (AB)$, then $A * X * B$ is true. By assumption, we have $A * B * C$ and therefore Proposition II.4 implies that $A * X * C$ is true, so that $X \in (AC)$. [*Note:* We are actually using an alternate form of this result; namely, $W * U * T$ and $W * V * U$ imply $W * V * T$. However, this follows from the stated form — $T * U * W$ and $U * V * W$ imply $T * V * W$ because $P * Q * R$ and $R * Q * P$ are equivalent conditions.]

B5. Both of the rays $[AB$ and $[BA$ are contained in the line AB, so we have $[AB \cup [BA \subset AB$. Conversely, suppose that $X \in AB$, and write $X = A + t(B - A)$ for some scalar t. If $t \neq 0$ then $X \in [AB$, while if $t < 0$ then we have $X * B * A$, and in fact we also have

$$X \;=\; B \;+\; (1-t)(B-A)\;.$$

Since $t < 0$, it follows that $1 - t > 1$ and therefore $X \in [BA$.■

B6. Since $A * B * C$ holds, we know that $C = A + v(B - A)$ where $v > 1$.

If $X \in [AB]$, then $X = A + t(B - A)$ where $0 \leq t \leq 1$ and hence $X \in [AB$. If $X \in [BC$, then $X = B + s(C - B)$ where $s \geq 0$; using the equation in the preceding paragraph, we may use this to rewrite X as a linear combination of A and B as follows:

$$X \;=\; B + s[A + v(B-A) - B] \;=\; (1+vs-s)B + (s-vs)A \;=\; A + (1+vs-s)(B-A)$$

Since $s \geq 0$ and $v > 1$, it follows that $1 + s - vs > 1$, and therefore we see that $X \in [AB$. Hence $[AB] \cup [BC$ is contained in $[AB$.

Conversely, suppose that $X \in [AB$ and write $X = A + t(B - A)$ where $t \geq 0$. If $t \leq 1$, then we know that $X \in [AB]$. Suppose now that $t > 1$. By the equation in the first paragraph we have

$$A \;=\; \frac{1}{1-v}\,C \;+\; \frac{-v}{1-v}\,B$$

and therefore after substitution and some algebraic calculation we may rewrite X as a linear combination of B and C as follows:

$$X \;=\; B \;+\; \frac{1-t}{1-v}\,(C-B)$$

Since $t, v > 1$ it follows that the numerator and denominator of $(1-t)/(1-v)$ are both negative, so that the quotient is positive, and therefore it follows that X must lie on $[BC$. Hence we have $[AB \subset [AB] \cup [BC$, and if we combine this with the previous paragraph we conclude that $[AB = [AB] \cup [BC$.■

B7. We shall follow the hint and eliminate all of the alternatives. In both cases the points X and Y are on M but not equal to A, and since L and M can only have the point A in common it follows that neither X nor Y lies on L. Therefore in each case either X and Y lie on the same side of L or else they lie on opposite sides of L.

For part (a), we are given that $A * X * Y$, and we want to show that X and Y cannot lie on opposite sides of L. However, if they did, then there would be some point C such that $C \in L$ and $X * C * Y$. Now C would have to be a point of M, and since A is the only common point of L and M it would follow that $A = C$, so that $X * A * Y$. However, we know that $A * X * Y$, and thus we cannot have $X * A * Y$. This is a contradiction, and the source is our assumption that X and Y were on opposite sides of L; hence they must be on the same side of L.■

For part (b), we are given that $X * A * Y$, and we want to show that X and Y cannot lie on the same side of L. But if they did, then by convexity all points of (XY) would also lie on that half-plane, and we know that $A \in (XY) \cap L$ does not. This is a contradiction, and the source is our assumption that X and Y were on the same side of L; hence they must be on opposite sides of L.■

B8. We first observe that all points of $(AC$ lie on a common side of AB, and likewise for $(BD$. If $X \in (AC$, then either $X = C$, $A * C * X$ or $A * X * C$ holds. In each case X lies on the same side of AB as C. The proof for $(BD$ can be obtained by replacing A and C with B and D respectively.

By assumption, C lies on one side of AB, say H, and D lies on the other, say K. We can now use the preceding paragraph to conclude that $(AC \subset H$ and $(BD \subset K$. Since H and K have no points in common, the same is true for $(AC$ and $(BD$. Furthermore, since AC meets AB in A and BD meets AB in B, it follows that A cannot lie on $[BD$ and B cannot lie on $[AC$. If we combine the conclusions of the preceding two sentences, we see that $[AC$ and $[BD$ have no points in common.■

B9. We know that $[AB]$ is contained in $\Delta ABC \cap AB$. We shall follow the hint and show that if $X \in \Delta ABC$ but $X \notin [AB]$, then $X \notin AB$.

If $X = C$, then the conclusion follows because $C \notin AB$ by assumption. We are now left with the cases where $X \in (AC)$ or $X \in (BC)$; since the argument in the second case is the same as the argument in the first with A replaced by B, it is enough to show that $X \notin AB$ if $X \in (AC)$. If we did have $X \in (AC)$ and $X \in AB$, then it would follow that the line L containing A and X would be equal to AB. But A, X, C all lie on a single lie by assumption, and this line must be $L = AB$, which means that all three vertices of ΔABC would lie on L. This is a contradiction, and the source is our assumption that (AC) and AB have a point in common. Therefore (AC) and $[AB]$ do not have any points in common; as noted before the same conclusion will follow for (BC) and $[AB]$, and thus we see that no points of $[AC] - \{A\}$ or $[BC] - \{B\}$ can lie on the line AB, so that $\Delta ABC \cap AB$ must be equal to $[AB]$.■

B10. In each of these problems, we need to rewrite the line equation in the form $g(x, y) = 0$, and then we need to compare the signs of $g(X)$ and $g(Y)$.

(a) In this case we may take $g(x, y) = 9x - 4y - 7$. We have $g(3, 6) = -3 < 0$ and $g(1, 7) = -26 < 0$, so the two points lie on the same side of L.■

(b) In this case we may take $g(x, y) = 3x - y - 7$. We have $g(8, 5) = 12 > 0$ and $g(-2, 4) = -29 < 0$, so the two points lie on opposite sides of L.■

(c) In this case we may take $g(x, y) = 2x + 3y + 5$. We have $g(7, -6) = 1 > 0$ and $g(4, -8) = -11 < 0$, so the two points lie on opposite sides of L.■

(d) In this case we may take $g(x, y) = 7x + 3y - 2$. We have $g(0, 1) = 1 > 0$ and $g(-2, 6) = 2 > 0$, so the two points lie on the same side of L.■

SOLUTIONS TO ADDITIONAL EXERCISES INVOLVING CONVEX FUNCTIONS

Here are the solutions to the additional exercises in `cvxfcnexercises.pdf`.

K0. We shall approach for the proof of Theorem 1 in `convex-functions.pdf`.

Suppose that $(\mathbf{x}, u)$ and $(\mathbf{y}, v)$ are points of $K \times \mathbb{R}$ such that $u > f(\mathbf{x})$ and $v > f(\mathbf{y})$; we need to prove that

$$t\,u \;+\; (1-t)\,v \quad > \quad f\,(t\,\mathbf{x} \;+\; (1-t)\,\mathbf{y}\,)$$

for all $t \in (0,1)$. The hypotheses imply that the left hand side satisfies

$$t\,u \;+\; (1-t)\,v \quad > \quad t\,f(\mathbf{x}) \;+\; (1-t)\,f(\mathbf{y})$$

and the convexity of f shows that the right hand side is greater than or equal to $f\,(t\,\mathbf{x} \;+\; (1-t)\,\mathbf{y}\,)$. Combining these, we obtain the inequality in first sentence of the paragraph.■

K1. For each example it suffices to prove that the second derivative is nonnegative.

(i) The second derivative is $f''(x) = \cosh x$, which is positive for all x.■

(ii) The second derivative is $f''(x) = \sinh x$, which is nonnegative for all $x \geq 0$.■

(iii) The second derivative is $f''(x) = 2\,\sec^2 x\,\tan x$, which is nonnegative if $0 \leq x < \frac{1}{2}\pi$.■

(iv) The second derivative is $f''(x) = -\sin x$, which is nonnegative if $\pi \leq x \leq 2\,\pi$.■

(v) The second derivative is $f''(x) = 1/x^2$, which is positive for all $x > 0$. $\pi \leq x \leq 2\,\pi$.■

K2. The key to proving the first statement is the following result:

Lemma. *Let T be a linear transformation from $\mathbb{R}^n$ to itself, and let $E \subset \mathbb{R}^n$ be convex. Then the image $T[E]$ is also convex.*

Proof. Suppose that $\mathbf{u} = T(\mathbf{x})$ and $\mathbf{v} = T(\mathbf{y})$, where $\mathbf{x}, \mathbf{y} \in E$, and let $t \in [0,1]$. Then by the linearity of T we have

$$t\,\mathbf{u} \;+\; (1-t)\,\mathbf{v} \quad = \quad T\,(t\,\mathbf{x} \;+\; (1-t)\,\mathbf{y})$$

and since E is convex the expression inside the parentheses lies in E. Therefore the right hand side lies in $T[E]$ and hence the latter is convex.■

We can now prove (i) as follows. Let $T(z,w) = (z,-w)$. We know that $-f$ is convex if and only if the set $E = \{(z,w) \mid w \geq -f(z)\}$ is convex.

Suppose that f is concave, so that $-f$ is convex and E is convex. Then by the lemma we know that $T[E]$ is also convex. But $T[E] = \{(z,s) \mid s \leq f(z)\}$, so this set is convex if E is convex. Conversely, if $E' = T[E]$ is convex, we can check directly that $E = T[E']$, and hence E is convex if $T[E]$ is convex, in which case $-f$ is convex and f is concave.■

Turning to (ii), lef $f_1 = -f$, let g be the linear function such that $g(a) = f_1(a)$ and $g(b) = f_1(b)$ as in the proofs of Theorem 2 and Lemma 3. Then $-f_1$ is convex, and hence f is concave, if $f_1'' \geq 0$; since $f_1'' = -f''$, it follows that if $f'' \leq 0$ then f is concave.■

K3. Let $E(f) = \{(z,w) \mid w \geq f(z)\}$ and $E'(g) = \{(z,w) \mid w \leq g(z)\}$; then $E(f)$ is convex because f is convex, and $E'(g)$ is convex because g is concave. Therefore the intersection $E(f) \cap E'(G)$ — which is the set described in the problem — is convex because it is the intersection of two convex sets.■

K4. Note that the vertices of the ellipse are $(\pm a, 0)$ and $(0, \pm 1)$. The hint is true because $|y| \leq b$ (where $b \geq 0$) if and only if $-b \leq y \leq b$. It will suffice to verify that f is concave on the closed interval $[-a, a]$, for if this is true then $-f$ will be convex and we can apply the conclusion of the preceding exercise.

It is convenient to rewrite $f(x) = a^{-1}\sqrt{a^2 - x^2}$, for then we can check directly that $f'(x) = -a^{-1}\, x\,(a^2 - x^2)^{-1/2}$ and

$$f''(x) \quad = \quad -a^{-1}\left((a^2 - x^2)^{-1/2} \; + \; x^2\,(a^2 - x^2)^{-3/2} \right) \quad = \quad -a\,(a^2 - x^2)^{-3/2} \; .$$

This expression is nonpositive if $|x| \leq a$, and therefore f is concave for these values of x.■

K5. To prove (i), follow the hint, and let g be the linear function such that $g(a) = f(a)$ and $g(b) = f(b)$. In this case the difference function $h = g - f$ satisfies $h'' > 0$ everywhere on the open interval (a, b), and we need to prove that $h(x) > 0$ for all x in that open interval. Once again there is some point $C \in (a, b)$ such that $h'(C) = 0$. Since $h'' > 0$ this means that $h'(x) > 0$ for $x < C$ and $h'(x) < 0$ for $x > C$. The first inequality implies that $h(x) > 0$ if $x \in [a, C]$, so if $h(x_0) \leq 0$ for some $x_0 \in (a, b)$ then we must have $x_0 \in (C, b)$. Since $h' < 0$ on (C, b) we know that h is strictly decreasing on $[C, b]$ and hence $h(x_0) = 0$ implies $h(b) < 0$, which is a contradiction. Hence there is no point $x \in (a, b)$ such that $h(x) = 0$, which is what we wanted to prove.■

We can now prove (ii) using the methods employed to prove Lemma 4 and Theorem 5 in `convex-functions.pdf`. Let $\mathbf{x}$ and $\mathbf{y}$ be distinct points of K, and write $\mathbf{v} = \mathbf{y} - \mathbf{x}$ (hence $\mathbf{v}$ is nonzero). If $\varphi(t) = f(x + t\,\mathbf{v})$ for t in some open interval containing $[0, 1]$, then the reasoning in the cited document implies that $\varphi'' > 0$ everywhere, so that φ is strictly convex, and we then have

$$f\,(t\,\mathbf{y} \; + \; (1-t)\,\mathbf{x}\,) \quad = \quad \varphi(t) \quad = \quad \varphi\,(t \cdot 1 + (1-t) \cdot 0) \quad < \quad t\varphi(1) \; + \; (1-t)\varphi(0) \; .$$

By construction $\varphi(0) = f(\mathbf{x})$ and $\varphi(1) = f(\mathbf{y})$, so the strict convexity of f follows from substitution of these values into the right hand side of the display above.■

SOLUTIONS TO ADDITIONAL EXERCISES FOR II.1 AND II.2

Here are the solutions to the additional exercises in `triangle-exercises.pdf`. Illustrations to accompany these solutions are given in the online file

`trianglefigures.pdf`

in the course directory.

C1. Suppose first that X is one of the vertices A, B, C. In these cases the conclusion follows because $A \in AB \cap AC$ and each of these lines contains an infinite number of points on the triangle (namely, all points of $[AB]$ and $[AC]$ respectively), $B \in AB \cap BC$ and each of these lines contains an infinite number of points on the triangle (namely, all points of $[AB]$ and $[BC]$ respectively), and finally $C \in BC \cap AC$ and each of these lines contains an infinite number of points on the triangle (namely, all points of $[BC]$ and $[AC]$ respectively).

Suppose now that X is not a vertex. Without loss of generality we may assume that $X \in (AB)$, for the remaining cases where $X \in (BC)$ or $X \in (AC)$ follow by interchanging the roles of A, B, C in the argument we shall give.

If $X \in (AB)$ and $L = AB$, then clearly $X \in L$ and L contains infinitely many points of the triangle because it contains $[AB]$. From now on, suppose that $X \neq AB$.

If $L = XC$, then X and C lie on both L and the triangle; we claim that no other point of L satisfies these conditions. Suppose to the contrary that there is such a third point Y; there are three cases depending upon whether Y lies on AB, BC or AC. If $Y \in AB$, then both L and AB contain the distinct points X and Y, so that $L = XY$; but we are assuming that X, Y, C are collinear, and this contradicts our even more basic assumption that A, B, C are noncollinear (this is implicit in asserting the existence of ΔABC). Therefore the line XC only meets the triangle in two points.

Now suppose that $C \notin L$ and $L \neq AB$; we need to show that L and ΔABC have at most one other point in common besides X. By Pasch's Theorem there is a second point Y on L which lies on either (BC) or (AC); in either case there we claim that there is no third point in $L \cap \Delta ABC$. Since $X \in AB$ is not one of the vertices and the lines BC and AC meet AB in B and A respectively, it follows that X lies on neither of these lines. Therefore the line $L = XY$ meets ΔABC in two sides and cannot contain any of the vertices. If there were a third point, it would lie on one of (AB), (BC) or (AC). By Exercises II.2.8 we know that L cannot contain points of all three sides,, and if the third point were in (AB) it would follow that $L = AB$. On the other hand, the line L cannot contain X and two points from either (AC) or (BC), for in that case L would be equal to AC or BC and also contain $X \in (AB)$, so that L would also be equal to AB. Thus the existence of a third point leads to a contradiction if $L \neq AB, XC$, and hence no such point can exist, so that all lines through X except AB meet the triangle in two points.■

C2. The most difficult parts of this proof were done in the preceding exercise. Let $\mathbf{T}$ be equal to $\Delta ABC = \Delta DEF$. By the preceding exercise, since $\mathbf{T} = \Delta ABC$ we know that $\{A, B, C\}$ is the set $\mathbf{V}$ of all points X in $\mathbf{T}$ such that two lines through X contain at least three points of $\mathbf{T}$, and likewise $\{D, E, F\}$ is the set $\mathbf{V}$ of all points X in $\mathbf{T}$ such that two lines through X contain at least three points of $\mathbf{T}$. Therefore we have $\{A, B, C\} = \mathbf{V} = \{D, E, F\}$.■

C3. Let H_1 and H_2 denote the two half-planes associated to L. Then each of the points A, B, C lies on exactly one of the subsets L, H_1, H_2.

Before we split the argument into cases using the previous sentence, we make a general observation. Since $X \in L$ lies in the interior of ΔABC, by the Crossbar Theorem we know that $(BX$ and (AC) have a point Y in common. This point cannot be X because a point cannot lie in both the interior of ΔABC and one of the three sides AB, BC, AC (look at the definition of interior). Since $Y \in (BC$, it follows that either $B * X * Y$ or $B * Y * X$ is true; however, if the latter were true, then B and X would lie on opposite sides of the line $AY = AC$, contradicting the assumption on X. Therefore we must have $B * X * Y$.

We claim that all three vertices cannot lie in either H_1 or H_2. If they did, then by convexity the point Y in the preceding paragraph would also lie in the given half-plane, and similarly the point X would lie in this half-plane. Since $X \in L$ by assumption, this is impossible, and thus the three vertices cannot all lie on one side of L.

Next, we claim that at most one vertex lies on L. If, say, $A \in L$, then $L = AX$, and if either B or C were also on L we would have that $L = AB$ or AC, which in turn would imply that $X \in AB$ or AC, contradicting the condition that X lies in the interior of the triangle. The cases where $B \in L$ and $C \in L$ can be established by interchanging the roles of the three vertices in the preceding argument.

Suppose now that one vertex lies on L; we claim that the other two vertices must lie on opposite sides of L. Once again, it is enough to consider the case where $A \in L$, for the other cases will follow by interchanging the roles of the vertices. But if $A \in L$, then the Crossbar Theorem implies that L and (BC) have a point W in common (in fact, the open segment (BC) and open ray $(AX$ have a point in common), and therefore it follows that B and C lie on opposite sides of L. Furthermore, it follows that the line L meets the triangle in the distinct points A and W.

The only possibility left to consider is the case where no vertex lies on L; by the preceding discussion, we know that neither half-plane contains all three vertices, and thus two of the vertices are on one half-plane and one is on the other. As before, without loss of generality we may assume that A is on one side and B, C are on the other. But in this situation we know that the line L meets both (BC) and (AC). This completes the examination of all possible cases.■

C4. We shall follow the hint and solve for $\mathbf{x}$ in terms of $\mathbf{y}$. Since $\mathbf{y} = A\mathbf{x} + \mathbf{b}$ and A is invertible, it follows that $\mathbf{x} = A^{-1}(\mathbf{y} - \mathbf{b})$. If we substitute this into the defining equation for the plane, we see that

$$d = C\mathbf{x} = CA^{-1}(\mathbf{y} - \mathbf{b}) \quad \text{or equivalently} \quad CA^{-1}\mathbf{y} = d + CA^{-1}\mathbf{b}$$

which shows that $\mathbf{y}$ is defined by an equation of the form $P\mathbf{y} = q$, where $P = CA^{-1}$ and $q = d + CA^{-1}\mathbf{b}$.■

C5. First of all, we claim that C and D lie on opposite sides of the line AB. Since no three of A, B, C, D are collinear, it follows that neither C nor D lies on AB, so it is only necessary to prove that C and D cannot lie on the same side of AB. However, if they did, then the noncollinearity condition would imply that $(AC$ and $(AD$ are distinct open rays, and by the Trichotomy Principle it would follow that either D would lie in the interior of $\angle BAC$ or else C would lie in the interior of $\angle BAD$.

By the preceding paragraph and Plane Separation, we know that the open segment (CD) meets the line AB in some point E, and since $C * E * D$ is true it follows that E lies in the interior of $\angle CAD$. Now B, E, A are collinear; since B does not lie in the interior of $\angle CAD$ and $(BE$ does lie in the interior of this angle, it follows that we must have $E * A * B$.

By the Supplement Postulate for angle measurements we have

$$|\angle BAC| + |\angle CAE| = 180^\circ = |\angle DAB| + |\angle DAE|$$

and since E lies in the interior of $\angle CAD$ the Addition Postulate implies that

$$|\angle CAD| = |\angle CAE| + |\angle DAE| .$$

If we add the equations which follow from the Supplement Postulate we have

$$|\angle BAC| + |\angle CAE| + |\angle DAB| + |\angle DAE| = 360^\circ$$

and if we now use the remaining equation we may rewrite the left hand side of the latter as $|\angle BAC| + |\angle CAD| + |\angle DAB| = 360^\circ$, which is what we wanted to prove.■