SOLUTIONS TO EXERCISES FOR

MATHEMATICS 133 — Part 6a

Fall 2020

Here are solutions for the exercises in the following file:

`http://math.ucr.edu/~res/math133-2020/execises/math133exercises03b.f20.pdf`

1. (*a*) Let $p, q, s > 0$ such that $s > p$, and consider the following four points A, B, C, D:

$$(0,0)\ , \quad (p,q)\ , \quad (s,0)\ , \quad (p,-q)$$

These points (in the given order) form the vertices of a convex quadrilateral because the segments $[AC]$ and $[BD]$ meet at the point $X = (p,0)$; the betweenness stateents follow because $-p < 0 < p$ (hence $B * X * D$) and $0 < p < s$ (hence $A * X * C$). If we choose $s > 2p$ then $s - p > p$, so that $|AB|^2 = p^2 + q^2 < (s = p)^2 + q^2 = |BC|^2 = |CD|^2$. If we had a parallelogram then we would have $AB| = |CD|$, so the kite $ABCD$ cannot be a parallelogram.■

(*b*) If the kite is a parallelogram, then we have $|AB| = |BC|$, $|AB| = |CD|$ and $|AD| = |BC|$, so that all four sides have equal length. This means that the convex quadrilateral must be a rhombus. Conversely, every rhombus is a kite by the definition of the latter.■

(*c*) By the perpendicular bisector theorem, we know that AC is the perpendicular bisector of $[BD]$. Therefore we have the right triangle congruences $\Delta AXB \cong \Delta AXD$ and $\Delta CXB \cong \Delta CXD$. By the basic properties of areas we have

$$\text{area}(ABCD) \;=\; \text{area}(AXB) \;+\; \text{area}(BXC) \;+\; \text{area}(CXD) \;+\; \text{area}(DXA)$$

and by the preceding congruences the right hand side equals

$$\tfrac{1}{2}\left(|AX|\,|XB| \;+\; |BX|\,|XC| \;+\; |CX|\,|XD| \;+\; |AX|\,|XD|\right)\ .$$

By the first sentence of this argument we have $|BX| = |XD| = \frac{1}{2}\,|BD|$. If we combine this with $|AC| = |AX| + |XC|$ (since $A * X * C$), the second formula for the area simplifies to $\frac{1}{2}\,|AC| \cdot |BD|$.■

2. There are five statements to prove, but everything is symmetric in x and y, so we can simlify the argument by considering pairs of statements,

The lines AB and EA are the x- and y-axes respectively, and in these cases it suffices to check the following:

(*i*) The first coordinates of C, D, E are all positive.

(*ii*) The second coordinates of B, C, D are all positive.

All of these are true by construction.

Now consider the lines BC and DE, which are defined by the equations $y = x - p$ and $x = y - p$ respectively. In these cases it suffices to check the following:

(*i*) If $f(x,y) = y - x + p$, then $f(D)$, $f(E)$ and $f(A)$ are all positive.

(*ii*) If $g(x,y) = x - y + p$, then $g(A)$, $g(B)$ and $g(C)$ are all positive.

These also follow from the definitions: $A = (0,0)$, $B = (p,0)$, $C = (p+q,q)$, $D = (q,p+q)$ and $E = (0,p)$.

Finally, consider the lines CD, which is defined by the equation $x + y = p + 2q$. In this case it suffices to check the following:

If $h(x,y) = x + y - p - 2q$, then $h(A)$, $h(B)$ and $h(E)$ are all negative.

As before, this statement can be checked directly.■

3. Let X be a point in the interior of $\angle ABC$, and let Y and Z denote the feet of the perpendiculars from X to AB and BC respectively. Since $|\angle ABC| < 90°$, it follows that both $\angle ABX$ and $\angle XBC$ are acute, and therefore by Theorem III.4.6 we know that $Y \in (BA$ and $Z \in (BC$. The condition in the problem now reduces to the equation

$$\frac{\sin|\angle ABX|}{\sin|\angle XBC|} = \frac{1}{2} .$$

Suppose that X satisfies this equation and $W \in (BX$. If U and V are the the feet of the perpendiculars from W to AB and BC respectively, then the right triangles ΔUBW and ΔWBV are similar to ΔYBX and ΔXBZ respectively by the hypotenuse-angle similarity theorem for right triangles. Since $\angle ABX = \angle ABW$ and $\angle XBC = \angle WBC$, it follows that

$$\frac{\sin|\angle ABW|}{\sin|\angle WBC|} = \frac{1}{2}$$

is also true. Therefore if X belongs to the set of interest to us, then this set also contains the entire ray $(BX$.

The preceding shows that the set of interest of us is a union of open rays (but this family of open rays may be empty or contain several rays). We need to show that there is only one open ray in this union. If $\theta = |\angle ABC|$, this amounts to showing that there is exactly one value of z such that $0 < z < \theta (< \frac{1}{2}\pi)$ and

$$\frac{\sin z}{\sin(\theta - z)} = \frac{1}{2} .$$

We shall do this by analyzing the left hand side as a function of z; denote this function by $h(z)$. The first thing to notice is that this function goes to 0 as $z \to 0$ and to $+\infty$ as $z \to \theta$. Next, we claim that this function is strictly increasing. The numerator is a positive valued strictly increasing function, and the denominator is a positive valued strictly decreasing function, so the quotient of the numerator by the denominator is a strictly increasing function. It follows that for each positive real number r there is a unique value of φ such that $0 < \varphi < \theta$ and $h(\varphi) = r$. In particular, if $r = \frac{1}{2}$ there is a unique value φ_0 such that $h(\varphi_0) = \frac{1}{2}$. It follows that if X belongs to the set, then it must belong to the unique ray $(BT$ such that $|\angle TBA| = \varphi_0$ (in radian measure), and hence the locus described in the problem is equal to the open ray $(BT$.■

4. Since the two rays are distinct and their open rays are contained in the same half-plane, the Protractor Postulate implies that $|\angle CAB| \neq |\angle DAB|$. This is inconsistent with both $\Delta CAB \sim \Delta DAB$ and $\Delta CBA \sim \Delta DBA$.■

5. By the results in Section III.6, there is a third number c such that there a triangle whose sides have lengths equal to a, b, c if and only if the three conditions in the Triangle Inequality are satisfied:

$$a \ < \ b + c \ , \quad b \ < \ a + c \ , \quad c \ < \ a + b$$

Since $a \leq b$ and $c > 0$ the first inequality is automatically satisfied. The seond inequality is equivalent to $b - a < c$, and therefore the condition for the three numbers to be the lengths of the sides for some triangle are $b - a < c < a + b$.■