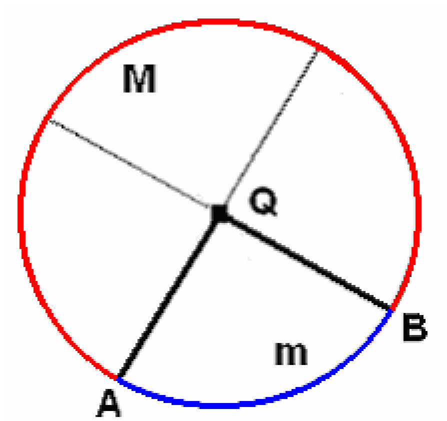
FIGURES FOR SOLUTIONS TO SELECTED EXERCISES

V : Introduction to non – Euclidean geometry

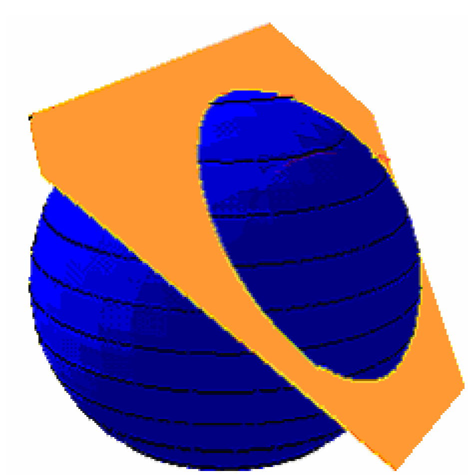
V.1 : Facts from spherical geometry

V.1.1.



The objective is to show that the minor arc **m** does not contain a pair of antipodal points but the major arc **M** does contain such a pair. As indicated by the hint, the minor arc consists of the endpoints and all points of the circle which lie in the interior of $\angle$**AQB.**

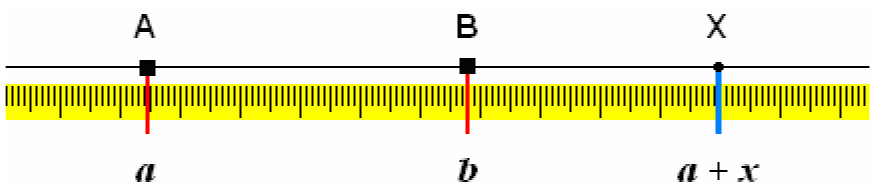
V.1.2.



The plane and the sphere intersect in a circle, and the objective of the problem is to find the center of the circle determined by the sphere with equation $x^2 + y^2 + z^2 = 1$ and the plane with equation $x + y + z = 1$. As noted in the hint, the circle's center is the foot of the perpendicular from the sphere's center to the plane.

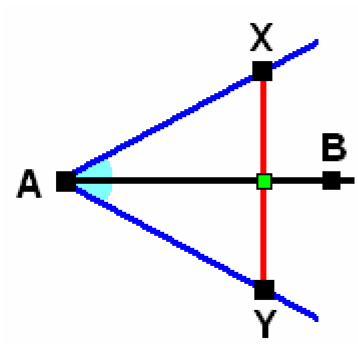
V.2 : Attempts to prove Euclid's Fifth Postulate

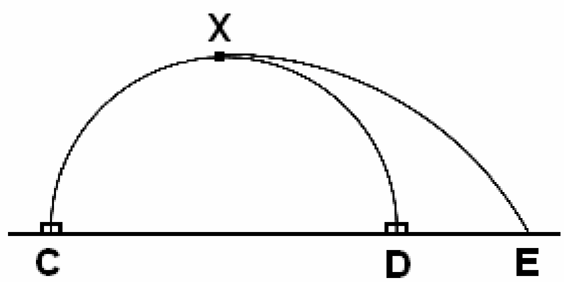
V.2.1(*a*).



Given the line **L** = **AB** and a ruler function **f** : **L** ⟶ $\mathbf{R}$, suppose that a = **f(A)** and b = **f(B)** satisfy $a < b$. Let $x > 0$, and consider the unique point **X** with **f(X)** = $a + x$. The objectives are to show that **X** lies on **(AB** and is the unique such point satisfying d(**A, X**) = x. .

V.2.1(*b*).

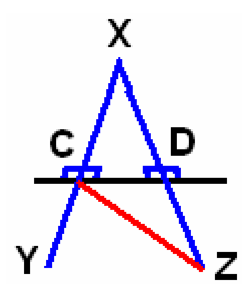




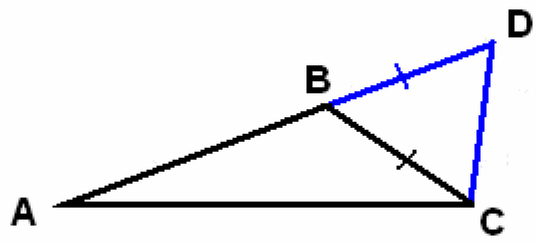
The figure on the left relates to the existence proof. As in Exercise **II.3.9,** one can construct △**YAB** such that **X** and **Y** lie on opposite sides of **AB** and △**XAB** ≅ △**YAB.** The objective is to prove that **XY** is perpendicular to **AB.**

The figure on the right relates to the first uniqueness proof. One has a hypothetical situation where there are two perpendiculars to the horizontal line which pass through **X,** and the goal is to show that this is impossible using the Exterior Angle Theorem.

The second uniqueness proof uses the following hypothetical picture, in which **C** and **D** are the midpoints of **[XY]** and **[XZ],** and this time the goal is to derive the impossible conclusion that |∠**ADC**| = |∠**ACD**| = **90°:**

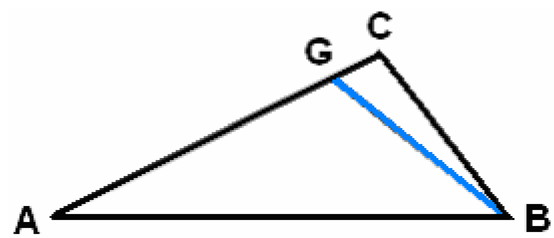


V.2.1(c).



Consider the triangle $\triangle$**ACD** such that **A**$*$**B**$*$**D** and d(**A, D**) = d(**A, B**) + d(**B, C**); it suffices to prove that the left hand side is greater than d(**A, C**), and the latter is equivalent to showing that |$\angle$**ACD**| > |$\angle$**ADC**|.

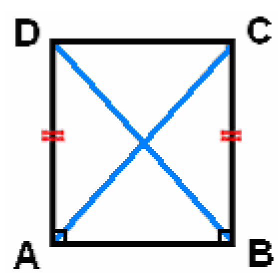
V.2.1(d).



It suffices to consider the case where d(**A, C**) > d(**D, F**); if equality holds then the conclusion is true by **SAS**, and if the reverse inequality holds one can reverse the roles of the two triangles in the argument for the given case. As suggested in the hint for this problem, consider the triangle $\triangle$**ABG** such that **G** $\in$ **(AC** and $\triangle$**ABG** $\cong$ $\triangle$**DEF.**

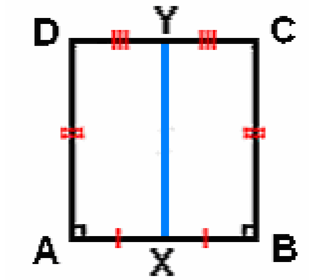
V.3 : Neutral geometry

V.3.3.



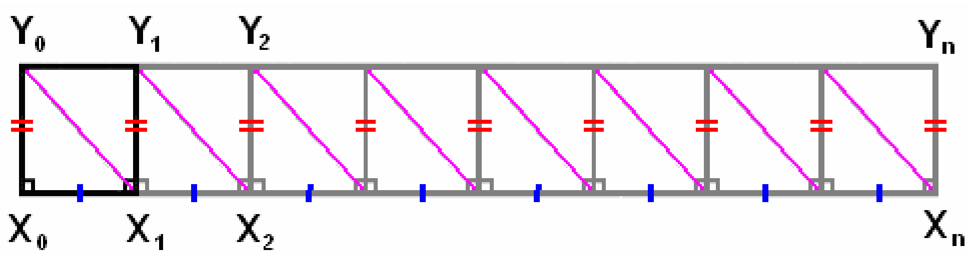
Split the Saccheri quadrilateral into two triangles two ways using the two diagonals. The idea of the proof is to show first that the two right triangles containing the base are congruent, and to use this fact as a step in proving that the two triangles containing the summit are congruent; equality of the summit angle measures will follow from this.

V.3.4.



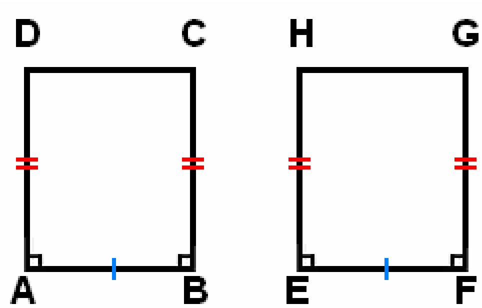
The idea is to prove that **X** is equidistant from **C** and **D** using congruent triangles (hence **XY** is the perpendicular bisector of **[CD]**) and to prove that **Y** is equidistant from **A** and **B** using congruent triangles (hence **XY** must also be the perpendicular bisector of **[AB]**). By the preceding exercise**,** the measures of the angles with vertices **C** and **D** are equal**.**

V.3.5.



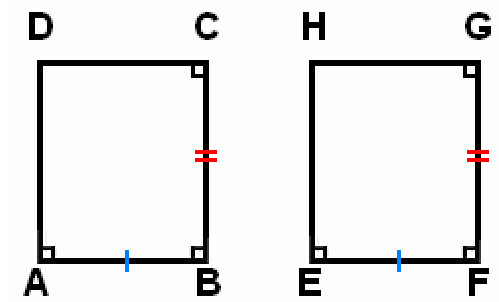
We shall assume the setting in the hint for this exercise**.** A major step in the proof is to show that the summits of the adjacent Saccheri quadrilaterals in the drawing have equal length**,** and this requires the use of auxiliary diagonal segments as in the picture above**.** It will then follow (by an induction argument) that the summits of all the Saccheri quadrilaterals in the picture will have equal lengths**.**

V.3.6.



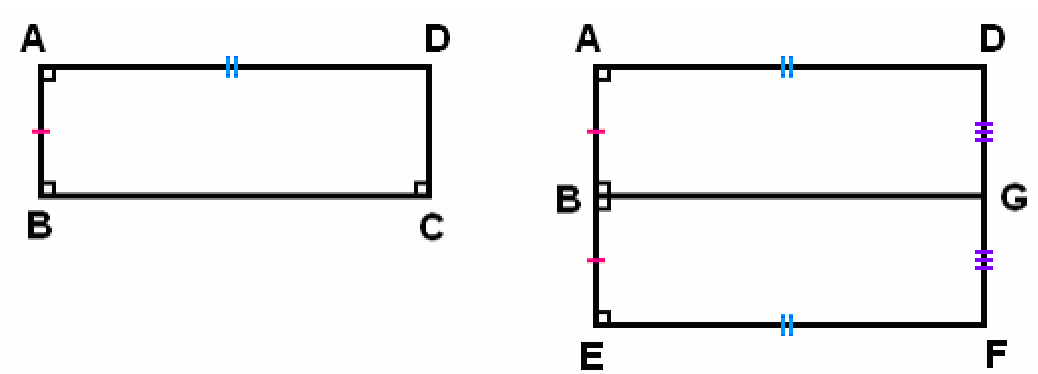
We are given a pair of Saccheri quadrilaterals as illustrated**.** One approach to proving this exercise is to split the quadrilaterals into two triangles along diagonals and show that the corresponding triangles are congruent**.**

V.3.7.



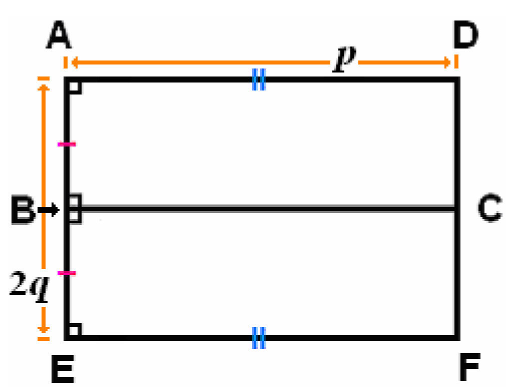
We are given a pair of Lambert quadrilaterals as illustrated. One approach to proving this exercise is to split the quadrilaterals into two triangles along diagonals and show that the corresponding triangles are congruent.

V.3.9.



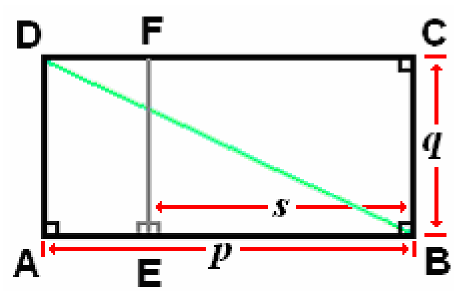
We begin with the Lambert quadrilateral ☐ **ABCD,** and we then construct a Saccheri quadrilateral ☐ **AEFD** such that the base of the latter has twice the length of d(**A, B**) and the lengths of the lateral sides are equal to d(**A, D**). The point **G** is the midpoint of the segment **[DF],** and the drawing suggests that **G = C.** Proving this is a key step in the argument. Observe that **BC** and **BG** are both perpendicular to **AB,** while **DC** is perpendicular to **BC** and **DF** is perpendicular to **BG.** By a previous exercise we have a relationship between d(**A, E**) and d(**D, F**).

V.3.11.



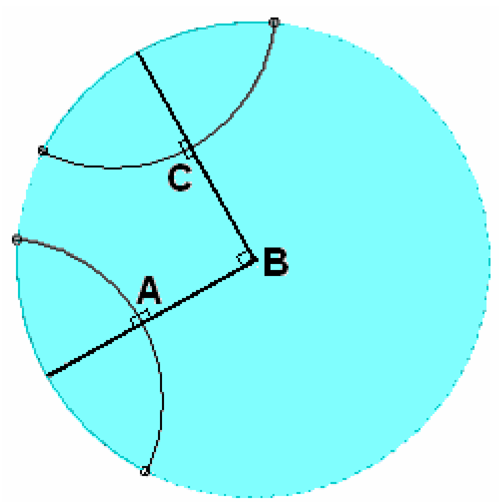
We know it is possible to construct a Saccheri quadrilateral whose base has length $2q$ and whose sides have length p. If we join the midpoints of the summit and base by a line segment, then by a previous exercise we obtain a pair of Lambert quadrilaterals, and the lengths of the appropriate sides are p and q.

V.3.12.



One has perpendicular lines as marked, and d(**B, E**) = s. The objective is to show that the perpendicular to **AB** at **E** contains a point **F** of **(CD);** if such a point exists, then **BC** and **EF** have a common perpendicular line **EB,** and also by construction the two lines **AB = EB** and **CD = CF** are both perpendicular to the line **BC.**

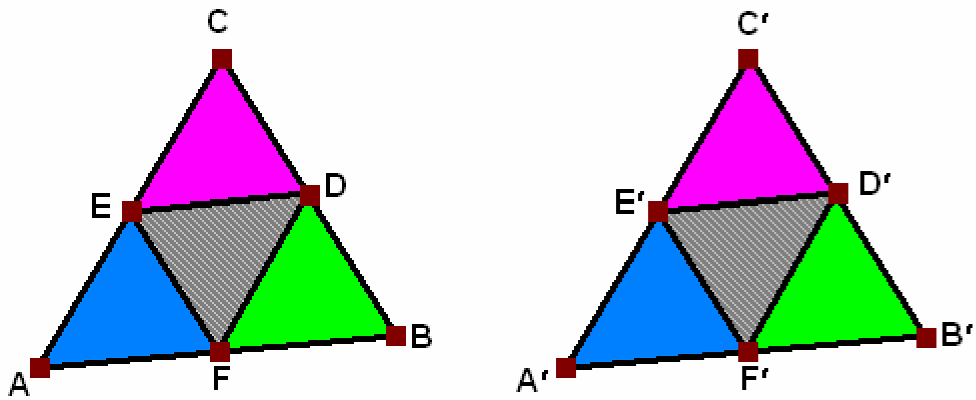
> **<u>Note.</u>** The drawing below provides an example illustrating the remarks following the statement of the exercise. It is given using the Poincaré model for hyperbolic geometry, which is described in Section **V.7** of the notes (in this model the radii and the arcs through **A** and **C** are lines, with right angles as illustrated).



> ***<u>Explanation.</u>*** As noted above, the arcs and radii represent lines in the hyperbolic plane, and these lines are perpendicular at all indicated points of intersection. In this particular example distances from the point **B** to the points **A** and **C** are equal. If d(**A, B**) and d(**B, C**) are too large, then as shown above it might not be possible to find a point **D** such that the points **A, B, C, D** form the vertices of a Lambert quadrilateral. More

generally, in such cases it is not even possible to find an ordered set of four points **W, X, Y, Z** which form the vertices of a Lambert quadrilateral (in that order) with right angles at **W, X** and **Y** and measurements d(**W, X**) = d(**A, B**), d(**X, Y**) = d(**B, C**).

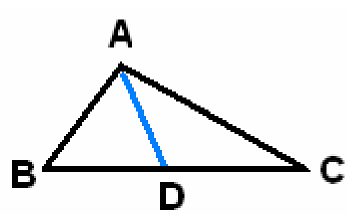
V.3.13. – V.3.14.



The points **D, E, F** and **D′, E′, F′** are the midpoints of the appropriate sides of △**ABC** and △**A′B′C′** respectively. In Exercise **V.3.13** the objective is first to prove that the three pairs of outside triangles with the same coloring are congruent and then to prove that the triangles in the middle of the two large triangles are congruent, while in Exercise **V.3.14** the objective is to show that all of the eight small triangles in the two large triangles are congruent to each other.

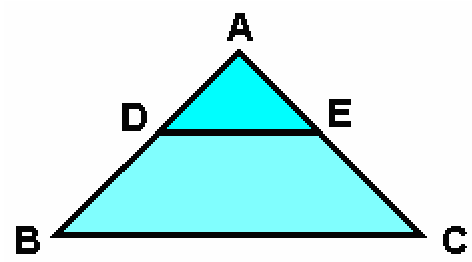
V.4 : Angle defects and related phenomena

V.4.3.



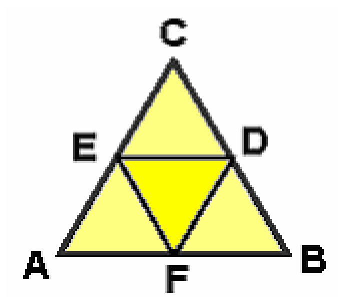
As indicated in the hint for this exercise, a preliminary objective is to show that the angular defect of one of the triangles △**ABD** and △**ABC** must be no greater than half the angular defect of △**ABC.** One way of proving the preliminary objective is to suppose this is false and use the additivity property of the angular defect to obtain a contradiction.

V.4.4.



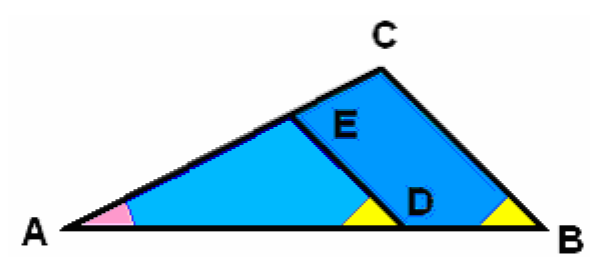
There are two isosceles triangles △**ABC** and △**ADE** with bases **[BC]** and **[DE]** such that **D** and **E** lie on the legs of △**ABC.** The angle defect of △**ADE** is less than the angle defect of △**ABC.**

V.4.5.



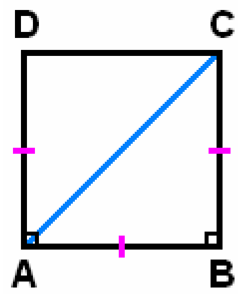
We know △**ABC** is equilateral, and the object is to prove the same for △**DEF,** given that **D, E, F** are the midpoints for the sides of △**ABC.** One plausibility argument for the assertion about angular measurements in the exercise is that the defect of the second triangle turns out to be less than the defect of the first, and for equilateral (hence equiangular) triangles the measure of the vertex angles ***increases*** as the defect ***decreases.*** Of course, the limiting value of the vertex angle measurement as the defect approaches $\mathbf{0}$ is equal to $\mathbf{60}$ degrees. For the final part of the problem, one step is to show that △**AEF** ≅ △**BDF** ≅ △**CDE** and to note that all these triangles are isosceles but not equilateral or equiangular. In fact, one can use the preceding exercise to compare |∠**ABC**| = |∠**CAB**| and |∠**AEF**|.

V.4.6.



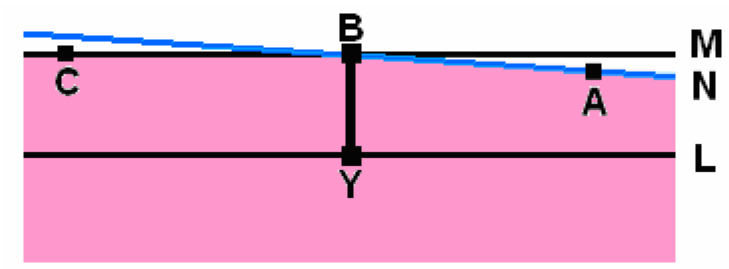
How do the defects of △**ABC** and △**ADE** compare, and what does this imply about the third angles of the two triangles**?**

V.4.7.



The Saccheri quadrilateral illustrated above is in a hyperbolic plane. What consequences can be drawn if the ray **[AC** bisects $\angle$**DAB?**

V.4.8.



The point **Y** is the foot of the perpendicular from **B** to **L,** the line **M** is perpendicular to the line **BY** (hence the latter is perpendicular to both **L** and **M**)**,** and **N** is a line through **B** which is parallel to **L** but distinct from **M** (such a second parallel line exists by our assumptions on **L** and **B**)**.** The angle $\angle$**ABC** is formed from one ray of **M** and one ray of **N** as suggested above**,** and it is given explicitly in the hint for the exercise**.**