Math 153
Spring 2016
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SOLUTIONS TO EXERCISES FROM `math153exercises11b.pdf`

All of the exercises considered here are *Additional exercises* which are in the cited exercise file. Drawings for the solutions are given on the pages at the end of this file.

4. Suppose that the fixed distance from the fixed line L is $d > 0$, and choose coordinate axes so that L becomes the x-axis. Then if P has coordinates (a, b) with $b \neq 0$, the argument in `locus-problems.pdf` shows that the distance minimum distance from P to L is $|b\,|$. Therefore (a, b) belongs to the locus if and only if $|b\,| = d$ or equivalently if and only if the second coordinate is $\pm\, d$. In other words, the locus consists of the two horizontal lines defined by the equation $y = \pm\, d$.■

5. Let r be the radius of the circle Γ, and choose coordinates so that the center of Γ is the origin; then the circle is defined by $x^2 + y^2 = r^2$. If (u, v) is the midpoint of a segment joining $(0, 0)$ to a point (p, q) on the circle Γ, then we have $u = \frac{1}{2}\,p$ and $v = \frac{1}{2}\,q$, so that (u, v) lies on the circle Γ^* of radius $\frac{1}{2}\,r$ centered at the origin. Conversely, if (u, v) lies on this circle then $u^2 + v^2 = \frac{1}{4}\,r^2$. Therefore if $(x, y) = (2u, 2v)$, then $x^2 + y^2 = r^2$, which means that (u, v) is the midpoint of a segment joining the center of the original circle Γ (namely, the origin) to the point (x, y), which lies on Γ.

To summarize, a point lies on the locus if and if it lies on the circle Γ^* whose center is the origin and whose radius is half the radius of Γ.■

6. Let r be the radius of the circle Γ, and choose coordinates so that the center Q of the circle is the origin, the line QY is the x-axis, and the first coordinate of Y is a positive number s; then $Y \notin \Gamma$ implies that $s \neq r$. The drawing in the file depicts two cases in which $s < r$ and $s > r$.

Since the function x^2 is strictly increasing for positive real numbers, it will suffice to show that the **square** of the distances is minimized if V lies on the line QY.

Our setting implies that the square of the distance from $Y = (s, 0)$ to a point $(r\,\cos\theta,\; r\,\sin\theta) \in \Gamma$ is given by

$$r^2\cos^2\theta\;-2rs\cos\theta\;+\;r^2\cos^2\theta\;+\;s^2\;\;=\;\;r^2\;-2rs\cos\theta\;+\;s^2$$

and in particular the square of the distance from Y to the point Z on $QY \cap \Gamma$ is $r^2 - 2rs + s^2$. If we subtract the square of the first distance from the square of the second, we see that the difference is $2rs(1 - \cos\theta)$. Now the expression inside the parentheses is zero if $\theta = 0$, in which case the point is Z, and it is positive if $0 < \theta < 2\pi$, in which case the point on Γ is not Z. Therefore the (square of the) distance from Y to a point on the circle Γ is minimized if and only if the latter point is Z.■

7. We shall solve this exercise assuming the weaker inequality $a < r$. Choose coordinates so that the center Q of the circle Γ is the origin.

Suppose that the distance from $X = (x, y)$ to Γ is equal to a; since the radius of Γ is equal to r, it follows that $X \neq Q$, so that $(x, y) \neq (0, 0)$. By the preceding exercise we know that X lies on the ray starting at Q and passing through the point on Γ whose coordinates are given by

$$\left(\frac{rx}{\sqrt{x^2 + y^2}} \, , \, \frac{ry}{\sqrt{x^2 + y^2}} \right) .$$

Furthermore, by the distance condition we know that the point X has coordinates

$$\left(\frac{tx}{\sqrt{x^2 + y^2}} \, , \, \frac{ty}{\sqrt{x^2 + y^2}} \right)$$

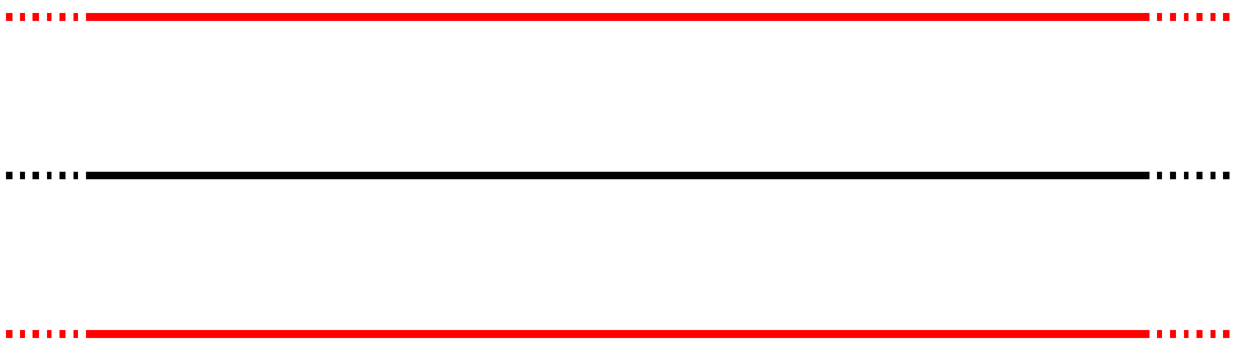
where $|t - r| = a$; *i.e.*, either $t = r - a$ or $t = r + a$. Therefore X either lies on the circle Γ_- with center Q and radius $r - a$ or on the circle Γ_+ with center Q and radius $r + a$.

Conversely, assume that X lies on either of these circles. Then by the preceding exercise the minimum distance from X to a point on the original circle Γ is given by a.■

DRAWINGS FOR THESE EXERCISES ARE ON THE NEXT TWO PAGES.

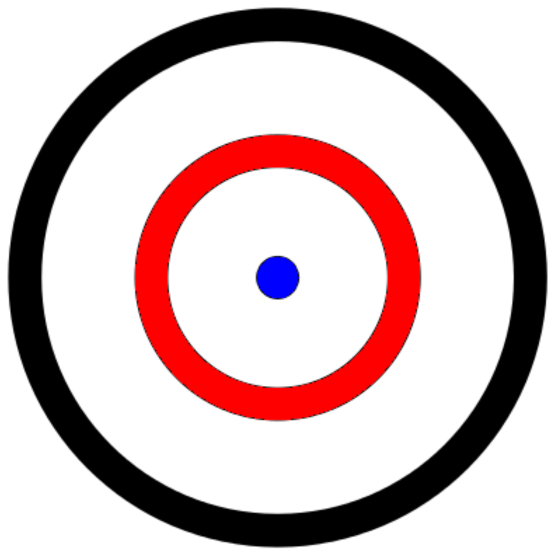
Drawings for the exercises

4.



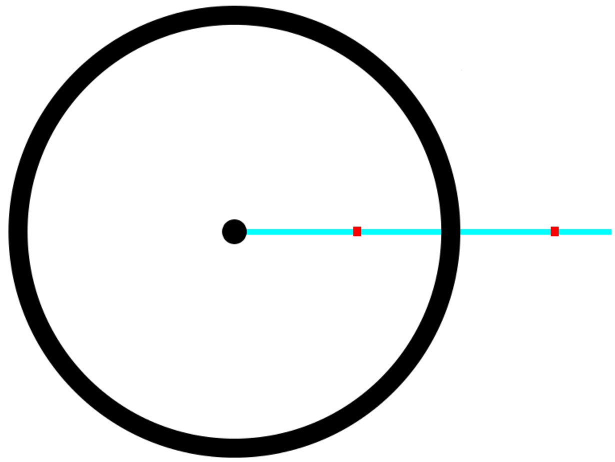
The original line is drawn in black, and the locus is drawn in red. The dots are meant to suggest that all three lines extend indefinitely to the left and to the right.

5.



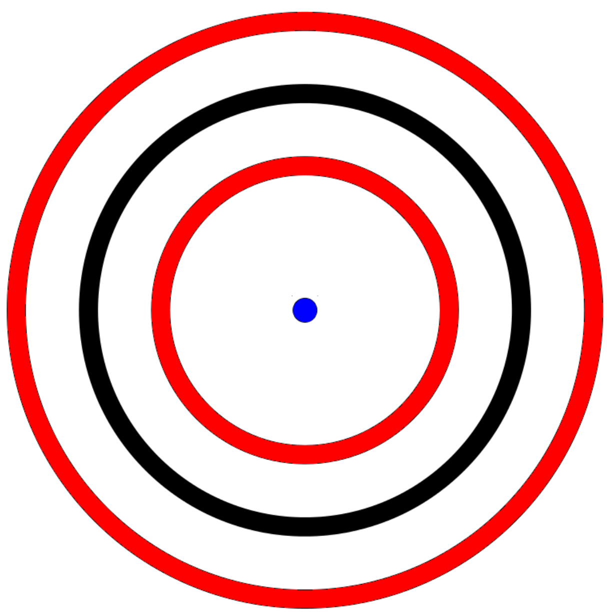
The original circle is drawn in black, the center of the circle is the blue dot, and the locus is drawn in red.

6.



The circle and its center are drawn in black. For each of the red points, the shortest distance to the circle lies along the blue line, which joins the red points to the center of the circle.

7.



The original circle is drawn in black, the center of the circle is the blue dot, and the locus is drawn in red.