Math 133
Fall 2018

UPDATED GENERAL INFORMATION — DECEMBER 6, 2018

Here is a summary of comments in yesterday's class with additional details.

The concurrence points of a triangle in Euclidean geometry

For right triangles, two of these points ae easy to find. The circumcenter is just the midpoint of the hypotenuse (recall a previous result stating that such a triangle is inscribed in a semicircle such that two of the vertices are endpoints of the semicircular arc) and the orthocenter is the vertex at which there is a right angle (if ΔABC has a right angle at B, then B is the foot of the perpendiculars from A to BC amd from C to AB, and by definition it lies on the altitude from B to AC).

Also, in an isosceles triangle ΔABC with $|AB| = |AC|$ and $|\angle BAC| > 90°$, one can see that both the circumcenter and orthocenter lie in the exterior of the triangle (*i.e.*, they are not on the triangle or in its interior). Experiment with examples to convince yourself of this fact.

Recognizing rectangles in neutral geometry

In order to understand the differences between Saccheri quadrilaterals in hyperbolic geometry and rectangles in neutral geometry, it is useful to understand how Saccheri quadrilaterals which have certain properties of rectangles must in fact be rectangles. The following was presented in class:

> *Suppose that we are working in a neutral plane and A, B, C, D are the vertices of a Saccheri quadrilateral with $AB, CD \perp BC$ and $|AB| = |CD|$. If in addition we know that the diagonals $[AC]$ and $[BD]$ bisect each other, then this quadrilateral is a rectangle.*

Since a Saccheri quadrilateral is a convex quadrilateral, we know that the diagonals intersect at some point E, and if we have a rectangle then we know that the diagonals do bisect each other.

A drawing is appended as the last page of this document. Recall that in a Saccheri quadrilateral the top (or summit) angles satisfy $|\angle ADC| = |\angle BCD|$ and the diagonals satisfy $|AC| = |BD|$.

The bisection condition yields the equations $|AE| = |EC| = \frac{1}{2}|AC| = \frac{1}{2}|BD| = |BE| = |ED|$, and by the Vertical Angle Theorem we have $|\angle AED| = |\angle BEC|$ and $|\angle AEB| = |\angle DEC|$. Furthermore, we know that each of the triangles

$$\Delta AED\ ,\quad \Delta BEC\ ,\quad \Delta AEB\ ,\quad \Delta DEC|$$

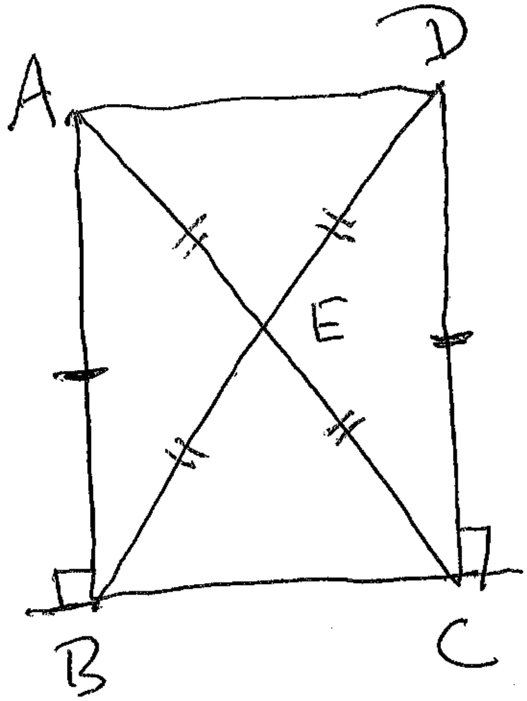
is isosceles by the bisection hypothesis. By **SAS** it follows that $\Delta AED \cong \Delta BEC$ and $\Delta AEB \cong \Delta DEC$. The congruence and isosceles properties now imply the following equations:

$$|\angle EAB| = |\angle EBA| = |\angle EDC| = |\angle ECD|\ ,\quad |\angle EAD| = |\angle EDA| = |\angle EBC| = |\angle ECB|$$

Since E is the midpoint of the two diagonals, the additivity property of angle measures now yields some additional equations:

$$|\angle BAD| = |\angle EAB| + |\angle EAD| = |\angle EBA| + |\angle EBC| = |\angle ABC| = 90°$$
$$|\angle CDA| = |\angle EDC| + |\angle EDA| = |\angle ECD| + |\angle ECB| = |\angle BCD| = 90°$$

Therefore the summit angles of the quadrilateral are also right angles and the Saccheri quadrilateral is in fact a rectangle.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