Equality in the Triangle Inequality

This document provides details for the approach taken in the lectures, which starts by answering the question for the real line:

Suppose that we are given three distinct points t_1, t_2 and t_3 on the real line. Under what conditions do we have $|t_3 - t_1| = |t_2 - t_1| + |t_3 - t_2|$?

Solution. There are six possible ways that the points can be ordered:

$$\begin{array}{ccccc} t_1 & < & t_2 & < & t_3 \\ t_1 & < & t_3 & < & t_2 \\ t_2 & < & t_1 & < & t_3 \\ t_2 & < & t_3 & < & t_1 \\ t_3 & < & t_1 & < & t_2 \\ t_3 & < & t_2 & < & t_1 \end{array}$$

We shall consider these cases in order.

If $t_1 < t_2 < t_3$ then $t_3 - t_1$, $t_2 - t_1$ and $t_3 - t_2$ are all positive so that

$$t_3 - t_1 = (t_3 - t_2) + (t_2 - t_1)$$

can be rewritten as $|t_3 - t_1| = |t_2 - t_1| + |t_3 - t_2|$. Therefore THE DISTANCES ADD IN THIS CASE.■

If $t_1 < t_3 < t_2$, then

$$|t_2 - t_1| = t_2 - t_1 > t_3 - t_1 = |t_3 - t_1|$$

which means that $|t_3 - t_1| = |t_2 - t_1| + |t_3 - t_2|$ cannot be true in this case and accordingly THE DISTANCES DO NOT ADD IN THIS CASE.■

If $t_2 < t_1 < t_3$, then by the preceding reasoning we have $|t_3-t_2| = |t_3-t_1|+|t_1-t_2| > |t_3-t_1|$ which means that THE DISTANCES DO NOT ADD IN THIS CASE.■

If $t_2 < t_3 < t_1$, then by the preceding reasoning we have $|t_1-t_2| = |t_1-t_3|+|t_3-t_2| > |t_3-t_1|$ which means that THE DISTANCES DO NOT ADD IN THIS CASE.■

If $t_2 < t_3 < t_1$, then by the preceding reasoning we have $|t_1-t_2| = |t_3-t_2|+|t_1-t_3| > |t_3-t_1|$ which means that THE DISTANCES DO NOT ADD IN THIS CASE.■

If $t_3 < t_2 < t_1$ then $t_3 - t_1$, $t_2 - t_1$ and $t_3 - t_2$ are all negative so that

$$t_3 - t_1 = (t_3 - t_2) + (t_2 - t_1)$$

can be rewritten as $-|t_3 - t_1| = -|t_2 - t_1| - |t_3 - t_2|$. The latter is equivalent to $|t_3 - t_1| = |t_2 - t_1| + |t_3 - t_2|$ Therefore THE DISTANCES ADD IN THIS CASE.■

To summarize, *the distances add if and only if either $t_1 < t_2 < t_3$ or $t_3 < t_2 < t_1$*.■

The general case of three collinear points

Assume now that x, y and z are collinear points in the coordinate plane $\mathbb{R}^2$. Then we know that

$$y \;\;=\;\; x \;+\; t(z-x)\;, \qquad \text{where} \quad t \in \mathbb{R}\;.$$

Then $|y-x| \;=\; |t(z-x)| \;=\; |t|\cdot|z-x|$ and similarly $|z-y| \;=\; |(1-t)(z-x)| \;=\; |1-t|\cdot|z-x|$.

Suppose now that $|z-x| = |y-x| + |z-y|$. If we substitute the values for the right hand summands in the previous paragraph and note that $|z-x| > 0$, we see that $1 \;=\; |t| + |1-t|$. Since $a = |b| + |c|$ and $a > 0$ imply $a > b$ and $a > c$, it follows that $t < 1$ and $1 - t < 1$. The latter is equivalent to $t > 0$, and therefore we have shown that if the inequality in the first sentence of this paragraph holds then $0 < t < 1$. — Conversely, if the latter holds then $1 \;=\; |t| + |1-t|$ and hence the reasoning of the preceding paragraph implies that $|z-x| = |y-x| + |z-y|$.■

The preceding duscussion also yields an alternate approach to part of the following result in `geometrynotes01.f13.pdf`: *If $|x+y| = |x| + |y|$ and $x, y \neq 0$ then x is a positive multiple of y and vice versa.* PROOF: *If $x, y \neq 0$ then $x = cy$ where $c > 0$, then $y = dx$ where $d > 1/c$ (so $d > 0$), and hence it suffices to to prove the first statement. As in the notes, by the Schwarz Inequality we know that x is a nonzero multiple of y, say $x = cy$; we need to show that $c > 0$. We have*

$$|1+c|\cdot|x| \;\;=\;\; |(1+c)x| \;\;=\;\; |x+cx| \;\;=\;\; |x+y| \;\;=\;\; |x| \;+\; |y| \;\;=$$

$$|x| \;+\; |cx| \;\;=\;\; |x| \;+\; |c|\cdot|x| \;\;=\;\; (1+|c|)\cdot|x|$$

which implies that $|1+c| \;=\; 1+|c|$. This equation holds if and only if $c \geq 0$, and since $c \neq 0$ it follows that $c > 0$.