Example of a geometric proof using vectors

Before starting the proof, we shall recall some results about linear independence. In general, a set S of vectors (finite for our purposes) is linearly independent if there is at most way of writing a given vector as a linear combination of the vectors in S. If S consists of only one vector, this amounts to saying that the vector is nonzero, and if S consists of two vectors this amounts to saying that neither vector is a scalar multiple of the other. In the following discussion, all vectors are assumed to lie in the coordinate plane $\mathbb{R}^2$.

Observe that the line L of points $(x, y) \in \mathbb{R}^2$ satisfying a nontrival first degree equation

$$ax + by = c$$

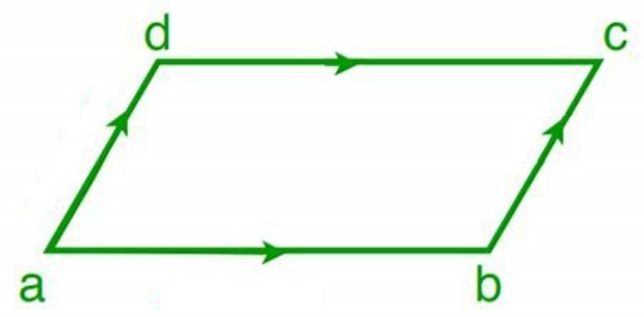
(where a and b are not both zero) has the form $\mathbf{v} + W$ where W is the 1–dimensional subspace of solutions to the *reduced equation*

$$ax + by = 0 .$$

Here is the result we wish to consider:

THEOREM. *The opposite sides of a parallelogram have equal length. In terms of vectors, if* $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, *and* $\mathbf{d}$, *are the vertices of the parallelogram, then* $\mathbf{b} - \mathbf{a} = \mathbf{c} - \mathbf{d}$ *and* $\mathbf{c} - \mathbf{b} = \mathbf{a} - \mathbf{d}$.

The following picture will probably be useful for understanding the formal proof.



Proof. The vector inequalities immediately yield the assertions about the lengths of the sides; this follows by taking the lengths of all the vectors in question. Also, it will suffice to prove the first vector equation. The second one will follow from the change of variables $\mathbf{a}' = \mathbf{b}$, $\mathbf{b}' = \mathbf{c}$, $\mathbf{c}' = \mathbf{d}$ and $\mathbf{d}' = \mathbf{a}$.

Since we are assuming that the figure under consideration is a parallelogram, it is not a line and therefore the vectors Furthermore, since the opposite edges of the parallelogram are parallel, we know that $\mathbf{d} - \mathbf{c} = p(\mathbf{b} - \mathbf{a})$ and $\mathbf{c} - \mathbf{b} = q(\mathbf{d} - \mathbf{a})$ for suitable nonzero scalars p and q; this is true because (1) the four vertices are distinct by assumption, (2) the lines **ab** and **cd** are parallel and the lines **bc** and **ad** are also parallel. In order to prove the theorem, we need to show that $p = q = 1$.

Consider the following identities:

$$\mathbf{d} - \mathbf{b} \;\; = \;\; (\mathbf{d} - \mathbf{c}) \; + \; (\mathbf{c} - \mathbf{b}) \;\; = \;\; -p(\mathbf{b} - \mathbf{a}) \; + \; q(\mathbf{d} - \mathbf{a})$$

$$\mathbf{d} - \mathbf{b} \;\; = \;\; (\mathbf{d} - \mathbf{a}) \; - \; (\mathbf{b} - \mathbf{a})$$

Now the vectors $\mathbf{d} - \mathbf{a}$ and $\mathbf{c} - \mathbf{a}$ are linearly independent, for if one were a multiple of the other then the points $\mathbf{a}$, $\mathbf{c}$, and $\mathbf{d}$ would be collinear and we know they are not by our assuption that the opposite sides are parallel. Therefore the preceding two equations yield

$$p(\mathbf{b} - \mathbf{a}) \; + \; q(\mathbf{d} - \mathbf{a}) \;\; = \;\; (\mathbf{d} - \mathbf{a}) \; - \; (\mathbf{b} - \mathbf{a}) \; .$$

By the linear independence of $\mathbf{d} - \mathbf{a}$ and $\mathbf{b} - \mathbf{a}$ we conclude that $p = q = 1$.■

Note that the preceding implies the standard Parallelogram Rule for sums of vectors. To see this, observe that the proof of the theorem yields the identity

$$\mathbf{c} - \mathbf{b} \;\; = \;\; \mathbf{d} - \mathbf{a}$$

so that $\mathbf{c} \; = \; \mathbf{b} + \mathbf{d} - \mathbf{a}$ or equivalently

$$\mathbf{c} - \mathbf{a} \;\; = \;\; (\mathbf{d} - \mathbf{a}) \; + \; (\mathbf{b} - \mathbf{a})$$

after subtracting $\mathbf{a}$ from both sides of the first identity.■