VERIFICATIONS OF THE SYNTHETIC AXIOMS IN COORDINATE GEOMETRY

This document refers repeatedly to the Mathematics 133 online notes `geometrynotes*.pdf` (where $*$ is one of `1`, `2a`, `2b`, `3a`, `3b`, `3c`, `4a`, `4b`, `5a`, `5b`, `5c`) in the following directory (which also includes this document):

`http://math.ucr.edu/~res/math133-2020`

There is a table of contents for these notes (which also describes the sections contained in each individual file) in `geometrycontents.pdf`, which is in the same directory.

As indicated in the notes mentioned above (see pp. 32–33), at some point it is necessary to do run a *relative consistency check* in order to verify that the synthetic axioms (or postulates) — which are stated in Unit II of the notes — actually hold for lines, planes, distances and angle measures as defined in coordinate geometry. Several parts of this work have been done in the notes, but a few others have not. Our purposes here are to give explicit references for the verifications which appear in the notes and to explain how one can finish the job.

The most difficult part turns out to be checking that the axioms for angle measurement hold, and the reason for this is that our definition of angle measurement involves the cosine function. In elementary mathematics courses this function is defined geometrically and many of its properties are shown by geometric arguments, but for the purposes of verifying the validity of the axioms we need to be able to define the cosine function and prove all its basic properties *without any* explicit *appeal to geometry*; if this is not done, the attempt to verify the axioms will probably be a circular argument.

Fortunately, it ****IS**** possible to define the cosine function and derive all its basic properties using the methods of differential calculus and infinite series; a totally rigorous proof that such an approach works would require material at a slightly higher level than this course, but it can be done if one has the background of a first real variables course (Mathematics 151A here). This will all be explained in an Appendix. The main body of the verification can be read if one assumes that *the cosine function, the sine function, and all of their basic properties can be established without* explicitly *appealing to geometry*. One can then view the Appendix as a formal justification of this assumption.

Incidence axioms

Strictly speaking, there are two cases to consider, depending upon whether we are working in two or three dimensions, but whenever possible we shall try to do things in a way which applies to both setting.

In the coordinate model for Euclidean geometry, the primitive concepts of *lines* and *planes* can be described in several equivalent ways. For our purposes it is convenient to think of lines in $\mathbb{R}^2$ or $\mathbb{R}^3$ as subsets of the form $\mathbf{v} + V$, where V is a 1-dimensional vector subspace of $\mathbb{R}^2$ or $\mathbb{R}^3$, and similarly it is convenient to think of planes in $\mathbb{R}^3$ as subsects of the form $\mathbf{v} + W$, where W is a 2-dimensional vector subspace. The discussion on pages 3–5 of

`http://math.ucr.edu/~res/progeom/pgnotes01.pdf` shows that these definitions are related to the ones which appear in elementary courses on coordinate geometry. At various points in our verification we shall need Lemma I.3.9 on page 21 of the notes.

The incidence axioms are stated on pages 35–36 of the notes; there is a short version for the two-dimensional case and a somewhat longer version for the three-dimensional case. For both cases, all the details are worked out on pages 13–15 of the following online file:

`http://math.ucr.edu/~res/progeom/pgnotes02.pdf`

The Euclidean Parallel Postulate (Playfair's Postulate)

This axiom does not require any primitive concepts beyond those which are needed for the incidence axioms. Playfair's Postulate is stated formally on page 77 of the notes, and a proof that it is true in $\mathbb{R}^2$ and $\mathbb{R}^3$ is given in Theorem II.14 on page 14 of the previously cited document `pgnotes02.pdf`.

Betweenness axioms

These axioms, which are stated on page 44 of the notes, require the additional primitive concept of **distance**, which is a function δ assigning to each ordered pair of points $(\mathbf{x}, \mathbf{y})$ a nonnegative real number $\delta(\mathbf{x}, \mathbf{y})$ which is zero if and only if the two points are qual and is symmetric in the two variables; in other words, we have $\delta(\mathbf{x}, \mathbf{y}) = \delta(\mathbf{y}, \mathbf{x})$. The conditions defining the 3-term betweenness relation $\mathbf{x} * \mathbf{y} * \mathbf{z}$ are that $\mathbf{x}, \mathbf{y}, \mathbf{z}$ must be distinct collinear points such that

$$\delta(\mathbf{x}, \mathbf{z}) \quad = \quad \delta(\mathbf{x}, \mathbf{y}) \ + \ \delta(\mathbf{x}, \mathbf{z}) \ .$$

Of course, in the coordinate model we take δ to be the usual Cartesian distance function

$$\mathbf{d}(\mathbf{x}, \mathbf{y}) \quad = \quad |\,\mathbf{x} - \mathbf{y}\,| \ .$$

By Theorem I.1.3 in the notes, the 3-term betweenness relation $\mathbf{x} * \mathbf{y} * \mathbf{z}$ holds if and only if

$$\mathbf{y} \quad = \quad \mathbf{x} \ + \ t(\mathbf{y} - \mathbf{x})$$

for some scalar t such that $0 < t < 1$.

It is fairly straightforward to verify both of the stated betweenness axioms.

(B–1) Given $\mathbf{b}$ and $\mathbf{d}$, one can check by direct computation of distances that the points $\mathbf{a} = \mathbf{b} - 2\mathbf{d}$, $\mathbf{c} = \frac{1}{2}(\mathbf{b} + \mathbf{d})$, and $\mathbf{e} = 2\mathbf{d} - \mathbf{b}$. satisfy $\mathbf{a} * \mathbf{b} * \mathbf{d}$, $\mathbf{b} * \mathbf{c} * \mathbf{d}$, and $\mathbf{c} * \mathbf{d} * \mathbf{e}$ respectively.

(B–2) Suppose that $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are distinct and collinear. Then se know that $\mathbf{c} = \mathbf{a} + t(\mathbf{b} - \mathbf{a})$ for some real number t where $t \neq 0, 1$ (if $t = 0$ then $\mathbf{c} = \mathbf{a}$ and if $t = 1$ then $\mathbf{c} = \mathbf{b}$). Once again, direct computation of distances shows that $\mathbf{c} * \mathbf{a} * \mathbf{b}$ holds if $t < 0$, while $\mathbf{a} * \mathbf{b} * \mathbf{c}$ holds if $0 < t < 1$ and $\mathbf{a} * \mathbf{b} * \mathbf{c}$ holds if $t > 1$.

We shall say more about the theorems on betweenness in Section II.2 of the notes when we discuss the linear measurement axioms.

Plane and Space Separation Postulates

No additional concepts are needed, but we do need the concept of convexity defined on page 48 of the notes and Proposition II.2.6 (the intersection of two convex sets is convex), which follows immediately from the definition and does not require any additional input. Statements of the separation postulates are given on page 49 of the notes.

The verification of the separation postulates proceeds in two steps:

(1) Verification of the Plane Separation Postulate for $\mathbb{R}^2$ and the Space Separation Postulate for $\mathbb{R}^3$.

(2) Verification of the Plane Separation Postulate for an arbitrary plane in $\mathbb{R}^3$ using the first step.

THE FIRST STEP. By the results on pages 3–5 of the file

`http://math.ucr.edu/~res/progeom/pgnotes01.pdf`

we know that lines in $\mathbb{R}^2$ and planes in $\mathbb{R}^3$ are given by equations of the form

$$\mathbf{a}\cdot\mathbf{x} \;=\; b$$

where b is a scalar and $\mathbf{a} \neq \mathbf{0}$. The two half-planes or half-spaces $H_\pm$ are defined to be the sets of points H_+ satisfying the inequality $\mathbf{a}\cdot\mathbf{x} > b$ and H_- satisfying the inequality $\mathbf{a}\cdot\mathbf{x} < b$. In order to verify the separation postulates, we need to prove the following:

(a) Each of the sets $H_\pm$ is nonempty and convex, and the $H_+ \cup H_- \cup F = \mathbb{R}^2$ or $\mathbb{R}^3$.

(b) If F is the given line or plane, then the sets $H_\pm$ and F are disjoint and there union is $\mathbb{R}^2$ or $\mathbb{R}^3$.

(c) If $\mathbf{p} \in H_+$ and $\mathbf{q} \in H_-$, then there is some $\mathbf{x} \in F$ such that $\mathbf{p} * \mathbf{x} * \mathbf{q}$.

We begin by verifying (a). To prove that the sets in question are nonempty, let $\mathbf{x} \in F$ and consider the points $\mathbf{x} \pm \mathbf{a}$. We then have

$$\mathbf{a}\,\cdot\,(\mathbf{x} \pm \mathbf{a}) \;=\; (\mathbf{x}\cdot\mathbf{a}) \,\pm\, |\mathbf{a}|^2 \;=\; b \,\pm\, |\mathbf{a}|^2$$

so that $\mathbf{x}+\mathbf{a} \in H_+$ and $\mathbf{x}-\mathbf{a} \in H_-$ and hence each of the sets $H_\pm$ is nonempty.

To prove that each set is convex, suppose first that $\mathbf{p}, \mathbf{q} \in H_+$, so that $\mathbf{a}\cdot\mathbf{p}$ and $\mathbf{a}\cdot\mathbf{q}$ are both greater than b. If $\mathbf{x}$ is between $\mathbf{p}$ and $\mathbf{q}$, then there is some t such that $0 < t < 1$ and $\mathbf{x} = (1-t)\mathbf{p} + t\mathbf{q}$, which means that

$$\mathbf{a}\cdot\mathbf{x} \;=\; \mathbf{a}\cdot\Big((1-t)\mathbf{p}+t\mathbf{q}\Big) \;=\; (1-t)\big(\mathbf{a}\cdot\mathbf{p}\big) \;+\; t\,(\mathbf{a}\cdot\mathbf{q})$$

and this is greater than b if $\mathbf{a}\cdot\mathbf{p}$ and $\mathbf{a}\cdot\mathbf{q}$ are both greater than b (note that both t and $1-t$ are positive). Therefore $\mathbf{x}$ must lie in H_+. Similar considerations hold if $\mathbf{p}, \mathbf{q} \in H_-$, so that $\mathbf{a}\cdot\mathbf{p}$ and $\mathbf{a}\cdot\mathbf{q}$ are both less than b. In this case, if $\mathbf{x}$ is between $\mathbf{p}$ and $\mathbf{q}$, then an argument like the preceding one shows that if $\mathbf{x}$ is between $\mathbf{p}$ and $\mathbf{q}$, then $\mathbf{a}\cdot\mathbf{x}$ is also less than b. Therefore we have shown that both H_+ and H_- are convex.

We shall now verify (b). To see that the three sets are disjoint and their union of is all of $\mathbb{R}^2$ or $\mathbb{R}^3$, note that H_+, F, and H_- are the sets of points where $\mathbf{a}\cdot\mathbf{x}$ is greater than, equal to, or less than b; since exactly one of these conditions holds for a given point, it follows that the point lies in exactly one of the three given subsets.

Finally, we shall verify (c). Suppose that we have $\mathbf{p} \in H_+$ and $\mathbf{q} \in H_-$, so that

$$\mathbf{a}\cdot\mathbf{q} \;=\; d \;<\; b \;<\; c \;=\; \mathbf{a}\cdot\mathbf{p}\,.$$

If we set

$$t \quad = \quad \frac{b-d}{c-d}$$

then it follows that $0 < t < 1$ and if $\mathbf{x} = (1-t)\mathbf{q} + t\mathbf{p}$, then

$$\mathbf{a} \cdot \mathbf{x} \quad = \quad (1-t)d \; + \; tc \quad = \quad d \; + \; t(c-d) \quad = \quad d \; + \; \frac{b-d}{c-d}\,(c-d) \;\; b$$

so that $\mathbf{x} \in F$ and $\mathbf{p} * \mathbf{x} * \mathbf{q}$.

THE SECOND STEP. Suppose that P is a plane in $\mathbb{R}^3$, and let L be a line in P. Let $x \in P$, and write L and P as $\mathbf{x} + V$ and $\mathbf{x} + W$ where V is a 1-dimensional vector subspace contained in the 2-dimensional subspace W. Let $\mathbf{z}$ be some vector not in W, and let Q be the plane containing L and the line joining $\mathbf{x}$ to $\mathbf{x} + \mathbf{z}$. If $\mathbf{v}$ is a nonzero vector which spans V, then it follows that $Q = \mathbf{x} + U$, where U is the subspace spanned by $\mathbf{x}$ and $\mathbf{z}$. By the Space Separation Postulate we know that the complement of Q consists of two half-spaces $K_\pm$ which are disjoint and convex, and that if $\mathbf{c}$ belongs to one and $\mathbf{d}$ to the other, then the line segment joining these points must contain a point in Q.

We claim that the sets $H_\pm = K_\pm \cap P$ have all the properties stated in the PSP. First of all, we need to show that $H_\pm$ is nonempty. Let $\mathbf{w}$ be a vector that is in W but not in V. We claim that $\mathbf{x} + \mathbf{w}$ lies in P but not in Q. The first statement is true by construction, and the second follows because if the vector did lie in Q then we would have $\mathbf{w} \in U$, so that $W \subset U$ and hence $P \subset Q$. Since P and Q are planes, the latter would imply $P = Q$, and we know this is false.

Choose $\mathbf{a}$ and b such that Q is defined by the equation $\mathbf{a} \cdot \mathbf{v} = b$ where $\mathbf{a}$ is nonzero. Then by construction we have $\mathbf{a} \cdot \mathbf{x} = b$ and $\mathbf{a} \cdot (\mathbf{x} + \mathbf{w}) = b' \neq b$. Therefore $\mathbf{x} + \mathbf{w}$ lies in one of the sets $H_\pm = K_\pm \cap P$. We also know that $\mathbf{a} \cdot \mathbf{w} = b' - b \neq 0$. To find a point in the other subset, consider the point $\mathbf{x} - \mathbf{w}$. In this case we have

$$\mathbf{a} \cdot (\mathbf{x} - \mathbf{w}) \quad = \quad b - (b' - b) \; .$$

Since one of the numbers

$$\{\, b' = b + (b' - b), \;\; b - (b' - b) \,\}$$

is smaller than b and one is larger, it follows that one of the points $\mathbf{x} \pm \mathbf{w}$ lies in $H_+ = P \cap K_+$ and the other lies in $H_- = P \cap K_-$. The completes the proof that each set $H_\pm$ is nonempty.

The proof that each set $H_\pm$ is convex is much easier, for we know that $K_\pm$ is convex, and since planes are convex it follows that the intersections $H_\pm = P \cap K_\pm$ are also convex.

Finally, suppose that we are given $\mathbf{p} \in H_+$ and $\mathbf{q} \in H_-$. Since $H_\pm \subset K_\pm$, by the Space Separation Postulate it follows that there is some $\mathbf{x} \in Q$ such that $\mathbf{p} * \mathbf{x} * \mathbf{q}$. However, since $\mathbf{p}$ and $\mathbf{q}$ both lie in P, we also know that $\mathbf{x} \in P$, so that $\mathbf{x} \in P \cap Q$. Since L is a line which is contained in the distinct planes P and Q, it follows that $P \cap Q = L$ and $\mathbf{x} \in L$ are required. This completes the proof of the lemma.■

Linear Measurement Axioms

Predictably, we are taking the usual distance in $\mathbb{R}^2$ or $\mathbb{R}^3$ as the notion of distance in the coordinate models for Euclidean geometry. Then axioms **(D–1)** and **(D–2)** on page 57 of the notes are just the elementary properties of the usual distance noted on page 3 of the notes. The

Ruler Postulate, which is also stated on page 57 of the notes, is verified in Exercise II.3.1 (see the file `math133exercises2.pdf`).

The synthetic approach to betweenness and separation theorems. Given the emphasis of betweenness and separation properties in the course notes, it seems appropriate (and perhaps reassuring) to note that *the Axioms of Incidence, Betweenness, Separation and Linear Measurement suffice to prove all the results on betweenness and separation that are formulated in the first three sections of Unit* II. — Since we use vector methods at various points in Sections II.2 and II.3 in order to prove several key results more quickly, a verification of this will require a somewhat different approach which is more abstract. All of the details are worked out in Sections 3.4–3.6 and 4.1–4.5 of Moïse (see pages 60–87). **We shall need many of the results on betweenness and separation** in our discussions of the Angle Measurement Axioms and Congruence Axioms below.

Angular Measurement Axioms

The axioms in this group are the most difficult to verify because the definition of angle measurement in $\mathbb{R}^2$ and $\mathbb{R}^3$ involves the inverse cosine function $\mathrm{Arc}\cos x$ defined from $[0, \pi]$ to $[-1, 1]$. In the Appendix we explain why one can define this function purely in terms of calculus, independently of any geometric considerations (at least from a strict logical viewpoint), and why this function must have several important properties that one expects.

In our verification of the Angle Measurement Axioms, we are allowed to use all results obtained prior to the introduction of these axioms, and we are also allowed to use the following:

(1) All results which can be proven synthetically from the preceding axioms.

(2) All results whose proofs can be done entirely using linear algebra.

Similar considerations will hold in our discussion of the Congruence Axioms.

Given three noncollinear points $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, the angle $\angle\mathbf{abc}$ is defined on page 58 of the notes, and the coordinate definition of the angle measure $\mu(\angle\mathbf{abc}) = |\angle\mathbf{abc}|$, or more correctly $\mu(\mathbf{a}, \mathbf{b}, \mathbf{c})$, is the function valued in the interval $(0, \pi)$ which is given on page 34 of the notes. Two remarks on the definition are appropriate at this point.

(1) Although the definition depends *a priori* upon choosing a suitable triple of points on the angle, the angle measurement axiom **(AM–0)** will imply that $\mu(\mathbf{a}, \mathbf{b}, \mathbf{c})$ does not depend upon the choices of these points.

(2) The definition on page 58 of the notes gives the angle measure in terms of radians; in the setting of the notes we often write angle measures in terms of degrees, and one can do this using the standard formula

$$(degrees) \quad = \quad (radians) \times (180/\pi) \ .$$

We have already mentioned one axiom for angle measurement which states that $\mu(\mathbf{a}, \mathbf{b}, \mathbf{c})$ only depends upon $\angle\mathbf{abc}$. This axiom and the other three are stated on page 61 of the notes, and they can be summarized as follows:

(AM–0) The **Invariance Property**, which as noted above states that $\mu(\mathbf{a}, \mathbf{b}, \mathbf{c})$ only depends upon $\angle\mathbf{abc}$.

(AM–1) The **Supplement Postulate**, which states that the measures of two (geometrically) supplementary angles add up to π radians or $180°$.

(AM–2) The **Protractor Postulate**, which gives conditions for the existence and uniqueness of angles with a given measurement.

(AM–3) The **Additivity Postulate**, which states that if one splits an angle α into two other angles α_1 and α_2 using a ray in the interior of α, then $|\alpha| = |\alpha_1| + |\alpha_2|$.

We shall verify these statements in the listed order.

VERIFICATION OF **(AM–0).** It will be helpful to begin by writing the angle measure function formally as

$$\mu(\mathbf{a}, \mathbf{b}, \mathbf{c}) \quad = \quad \operatorname{Arc}\cos\left(\frac{(\mathbf{a}-\mathbf{b})\cdot(\mathbf{c}-\mathbf{b})}{|\mathbf{a}-\mathbf{b}|\cdot|\mathbf{c}-\mathbf{b}|}\right) \, .$$

We shall also need a property of angles which is probably clear intuitively but still must be verified logically.

CLAIM. In every angle $\angle\mathbf{xyz}$ there are exactly two maximal collinear subsets with more than two points; namely, the edges $[\mathbf{yx}$ and $[\mathbf{yz}$.

Verification. We shall only sketch the proof and leave it to the reader to fill in the details. By definition, the two edges are collinear sets. To see that $[\mathbf{yz}$ is a maximal collinear set, let $\mathbf{w}$ be a point which lies on the angle but not on this edge, so that $\mathbf{w} \in (\mathbf{yx}$. Then the lines $\mathbf{yx}$ and $\mathbf{yw}$ are the same, so if $\mathbf{w} \in \mathbf{yz}$ then we would have $\mathbf{yz} = \mathbf{yw} = \mathbf{yx}$, which contradicts the basic condition that $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ must be noncollinear in order to define $\angle\mathbf{xyz}$. Therefore $[\mathbf{xz}$ is a maximal collinear subset; we can now prove the same conclusion for $[\mathbf{yx}$ by switching the roles of $\mathbf{x}$ and $\mathbf{z}$ in the preceding argument.

To see that these are the only maximal collinear subsets, suppose that we are given some arbitrary maximal collinear set M, which we might as well assume contains at least two points $\mathbf{u}, \mathbf{v}$. For each point $\mathbf{p}$ in $\angle\mathbf{abc}$, exactly one of the three alternatives

$$\mathbf{p} \ \in \ (\mathbf{yx} \, , \qquad \mathbf{p} \ = \ \mathbf{y} \, , \qquad \mathbf{p} \ \in \ (\mathbf{yz}$$

must hold. Applying this to the points $\mathbf{u}, \mathbf{v}$, we see that exactly one of the following statements must be true:

$\mathbf{u}, \mathbf{v} \in [\mathbf{yz}$.

$\mathbf{u}, \mathbf{v} \in [\mathbf{yx}$.

One of $\mathbf{u}, \mathbf{v}$ lies in $[\mathbf{yz}$, and the other lies in $[\mathbf{yx}$.

In the first two cases we know that the maximal collinear set M must be contained in $\mathbf{yz}$ or $\mathbf{xz}$, and since $[\mathbf{yz}$ and $[\mathbf{xz}$ are maximal collinear subsets of the angle, it follows that M must be equal to the corresponding ray. In the remaining case, we claim that $\mathbf{u}, \mathbf{v}$ is a maximal collinear subset. It will suffice to show that the line $L = \mathbf{uv}$ meets the angle only at the two given points. As usual, it will suffice to consider the case where $\mathbf{u} \in (\mathbf{yx}$ and $\mathbf{v} \in (\mathbf{yz}$. If $\mathbf{w}$ were a third point in the intersection, then it would have to lie on (at least) one of the two lines $\mathbf{yx}$, $\mathbf{zx}$, and therefore L would have to be equal to one of these edge lines. On the other hand, we know that one of the points $\mathbf{u}, \mathbf{v}$ does not lie on this edge line (if it did, then both edges would be collinear). Therefore L and the angle have exactly two points in common.■

COMPLETION OF THE VERIFICATION OF OF **(AM–0).** If $\angle\mathbf{dbe} = \angle\mathbf{abc}$, then these angles have the same maximal collinear subsets, so that one of the following alternatives is true:

$$\text{Either } [\mathbf{bc} = [\mathbf{ef} \text{ and } [\mathbf{ba} = [\mathbf{ed}, \quad \text{or else} \quad [\mathbf{bc} = [\mathbf{ed} \text{ and } [\mathbf{ba} = [\mathbf{ef}$$

If we can complete the argument in one of these cases, then the other will follow by switching the roles of $\mathbf{d}$ and $\mathbf{f}$, so we shall assume that the first alternative holds.

First of all, we have

$$\mathbf{b} \quad = \quad [\mathbf{ba} \cap [\mathbf{bc} \quad = \quad [\mathbf{ed} \cap [\mathbf{ef} \quad = \quad \mathbf{e}\ .$$

Since $\mathbf{f} \in (\mathbf{bc}$ and $\mathbf{d} \in (\mathbf{ba}$, there are positive scalars s, t such that $\mathbf{f} - \mathbf{e} = s(\mathbf{c} - \mathbf{b})$ and $\mathbf{d} - \mathbf{e} = t(\mathbf{a} - \mathbf{b})$. If we substitute these expressions into the function μ, we obtain the following chain of equations:

$$\mu(\mathbf{d}, \mathbf{e}, \mathbf{b}) \quad = \quad \operatorname{Arc\,cos}\left(\frac{(\mathbf{d}-\mathbf{e})\cdot(\mathbf{b}-\mathbf{e})}{|\mathbf{d}-\mathbf{e}|\cdot|\mathbf{b}-\mathbf{e}|}\right) \quad =$$

$$\operatorname{Arc\,cos}\left(\frac{t(\mathbf{a}-\mathbf{b})\cdot s(\mathbf{c}-\mathbf{b})}{|t(\mathbf{a}-\mathbf{b})|\cdot|s\mathbf{c}-\mathbf{b}|}\right) \quad =$$

$$\operatorname{Arc\,cos}\left(\frac{st\cdot\big((\mathbf{a}-\mathbf{b})\cdot(\mathbf{c}-\mathbf{b})\big)}{st\big(|\mathbf{a}-\mathbf{b}|\cdot|\mathbf{c}-\mathbf{b}|\big)}\right) \quad =$$

$$\operatorname{Arc\,cos}\left(\frac{(\mathbf{a}-\mathbf{b})\cdot(\mathbf{c}-\mathbf{b})}{|\mathbf{a}-\mathbf{b}|\cdot|\mathbf{c}-\mathbf{b}|}\right) \quad = \quad \mu(\mathbf{a}, \mathbf{b}, \mathbf{c})$$

This completes the verification of the Invariance Property.■

VERIFICATION OF **(AM–1).** For this axiom we are given $\angle\mathbf{abc}$ and a point $\mathbf{d}$ such that $\mathbf{d} * \mathbf{b} * \mathbf{c}$ holds. Since the cosine function is strictly decreasing on $(0, \pi)$ and satisfies $\cos(\pi - \theta) = -\cos\theta$, it will suffice to prove that $\cos\angle\mathbf{abd} = -\cos\angle\mathbf{abc}$.

The betweenness hypothesis implies that $\mathbf{d} - \mathbf{b} = r(\mathbf{c} - \mathbf{b})$ for some $r < 0$. Therefore we have the following chain of equations:

$$\cos\angle\mathbf{abd} \quad = \quad \frac{(\mathbf{a}-\mathbf{b})\cdot(\mathbf{d}-\mathbf{b})}{|\mathbf{a}-\mathbf{b}|\cdot|\mathbf{d}-\mathbf{b}|} \quad = \quad \frac{(\mathbf{a}-\mathbf{b})\cdot t(\mathbf{c}-\mathbf{b})}{|\mathbf{a}-\mathbf{b}|\cdot|t(\mathbf{c}-\mathbf{b})|} \quad = \quad \frac{t\cdot\big((\mathbf{a}-\mathbf{b})\cdot(\mathbf{c}-\mathbf{b})\big)}{|t|\cdot|\mathbf{a}-\mathbf{b}|\cdot|(\mathbf{c}-\mathbf{b})|} \quad =$$

$$\frac{t}{|t|}\cdot\frac{(\mathbf{a}-\mathbf{b})\cdot(\mathbf{c}-\mathbf{b})}{|\mathbf{a}-\mathbf{b}|\cdot|\mathbf{c}-\mathbf{b}|} \quad = \quad \frac{t}{|t|}\cdot\cos\angle\mathbf{abc}$$

Since $t < 0$ it follows that $t/|t|$ is equal to -1, and therefore we have verified the equation $\cos\angle\mathbf{abd} = -\cos\angle\mathbf{abc}$ which is needed to establish **(AM–1).**■

VERIFICATION OF **(AM–2).** We shall use barycentric coordinates to verify the Protractor Postulate. In Unit I of the notes we only developed barycentric coordinates for points in $\mathbb{R}^2$, but one can also define barycentric coordinates in $\mathbb{R}^3$. Some exercises in the notes address this issue, and a complete discussion appears in Section 3 of the following previously cited reference:

`http://math.ucr.edu/~res/progeom/pgnotes02.pdf`

In order to make everything work in the three-dimensional case, we shall need the following generalization of Proposition II.2.9 from page 60 of the notes.

PROPOSITION II.2.9–$(n - \dim)$**.** *Suppose that A, B, C, D are points in $\mathbb{R}^n$ $(n \geq 2)$ such that C and D do not lie on the line AB but D lies on the unique plane P containing A, B, and C. Let $D = xA + yB + zC$ be the unique expression for D in terms of barycentric coordinates, so that $x + y + z = 1$. Then C and D lie on the same side of AB if z is positive, and they lie on opposite sides of AB if z is negative.*

In the course of proving this result we shall need the following observation (which is probably intuitively clear but nevertheless requires a formal proof).

LEMMA. *Suppose that L is a line in $\mathbb{R}^n$ and P is a plane in $\mathbb{R}^n$ such that $L \subset P$. Then there is only one way of writing $P - L$ as a union of two disjoint convex subsets satisfying properties $(a) - (c)$ in the discussion of the Plane Separation Postulate (as above).*

Proof. Suppose that we are given decompositions $H_1,\ H_2$ and $K_1,\ K_2$; then we may write $P - L$ as the union of the four pairwise disjoint subsets

$$H_1 \ \cap \ K_1 \ , \quad H_1 \ \cap \ K_2 \ , \quad H_2 \ \cap \ K_1 \ , \quad H_2 \ \cap \ K_2$$

and similarly we have

$$H_1 \quad = \quad (H_1 \cap K_1) \ \cup \ (H_1 \cap K_2) \ , \qquad H_2 \quad = \quad (H_2, \cap K_1) \ \cup \ (H_2, \cap K_2) \ .$$

In particular, it follows that at least one of the sets $H_1 \cap K_1$, $H_1 \cap K_2$ must be nonempty. We need to show that the other set must be empty. Suppose that both are nonempty, and let $\mathbf{x}_j \in H_1 \cap K_j$. Then by property (c) there is some point $\mathbf{y} \in L$ such that $\mathbf{x}_1 * \mathbf{y} * \mathbf{x}_2$ holds. Since H_1 is convex, it follows that $\mathbf{y} \in H_1$. However, we know that $H_1 \cap L = \emptyset$, so this is a contradiction. Therefore we see that one of the sets $H_1 \cap K_1$, $H_1 \cap K_2$ must be nonempty and the other must be empty. Suppose that the first intersection is nonempty and the second is empty. Then it follows that $H_1 \subset K_1$. Likewise, if the second intersection is nonempty and the first is empty. then it follows that $H_1 \subset K_2$. Switching the roles of H and K in the preceding discussion, we also see that if $H_i \cap K_j$ is nonempty, then $K_i \subset H_j$.

The reasoning in the preceding paragraph shows that if $H_i \cap K_j$ is nonempty, then we must have $H_i = K_j$. Furthermore, if we take m and n in $\{1, 2\}$ to be different from i and j respectively, then the description of H as a dijoint union of the four intersections implies that we must also have $H_m = K_n$.■

Proof of Proposition II.2.9–$(n - \mathrm{dim})$. Let $H_\pm$ be the sets of points in the plane P where z is positive or negative. We claim these satisfy the defining conditions for half-planes, and therefore by the preceding lemma they are the two half-planes.

The set H_+ contains C and therefore is nonempty, while the set H_- contains $A - 2C$ and therefore is also nonempty. Disjointness follows because the barycentric coordinate of C cannot be both positive and negative. To see convexity, suppose that $D_i = x_i A + y_i B + z_i C$, where $i = 1, 2$. If D_1 and D_2 belong to the same set $H_\pm$ then either $z_1, z_2 > 0$ or else $z_1, z_2 < 0$. in either case, suppose that $X = tD_1 + (1 - t)D_2$ where $0 < t < 1$. It follows that $1 - t$ is also positive, so that $tz_1 + (1 - t)z_2$ is nonzero with the same sign as z_1 and z_2. Therefore X lies on the set $H_\pm$ of AB as D_1 and D_2, and therefore each set $H_\pm$ must be convex.

Finally, suppose that one of the points D_1, D_2 lies in H_+ and the other lies in H_-. As usual, we might as well assume that $D_1 \in H_-$ and $D_2 \in H_+$. In order to verify property (c), we need to find some t such that $0 < t < 1$ and the barycentric coordinate of $X = tD_1 + (1 - t)D_2$ — which is $tz_1 + (1 - t)z_2$ — is equal to zero (note that $X \in AB$ if and only if this barycentric coordinate is equal to zero). In other words, we need to show that if t solves the equation $tz_1 + (1 - t)z_2 = 0$, then $0 < t < 1$.

Direct calculation shows that the solution to the equation in the preceding sentence is

$$t \quad = \quad \frac{z_2}{z_2 - z_1} \ .$$

Since $0 < z_2 < z_2 - z_1$ (remember that $z_1 < 0$), it follows that $0 < t < 1$ as required.■

COMPLETION OF THE VERIFICATION OF OF **(AM–2).** If are given three noncollinear points A, B, C, and a number θ such that $0 < \theta < 180$, then we want to show that there is a unique ray $[AD$ such that $(AD$ and C lie on the same side of AB and $|\angle DAB| = \theta$.

Define unit vectors $\mathbf{u}$ and $\mathbf{v}$ by the equations

$$\mathbf{u} \;=\; \frac{1}{|B-A|}\,(B\;-\;A)\;, \qquad \mathbf{v} \;=\; \frac{1}{|C-A|}\,(C\;-\;A)$$

and use the Gram-Schmidt process to construct a unit vector $\mathbf{w}$ such that $\mathbf{w} = p\mathbf{v} + q\mathbf{u}$ such that $\mathbf{w}$ is perpendicular to $\mathbf{u}$ and $p > 0$. Now let

$$\mathbf{y} \;=\; \cos\theta\,\mathbf{u} \;+\; \sin\theta\,\mathbf{v}$$

and define vectors E and F, in the plane of A, B and C as follows:

$$E \;=\; A + \mathbf{u}\,, \qquad F \;=\; A + \mathbf{y} \;=\; F + \cos\theta\,\mathbf{u} + \sin\theta\,\mathbf{w} \;=\; F + \cos\theta\,\mathbf{u} + \sin\theta\,(p\,\mathbf{v} + q\,\mathbf{u} \;=$$

$$\left(1 \;-\; \frac{\cos\theta + q\sin\theta}{|B-A|} \;-\; \frac{p\sin\theta}{|C-A|}\right) A \;+\; \frac{\cos\theta + q\sin\theta}{|B-A|}\,B \;+\; \frac{p\sin\theta}{|C-A|}\,C\,.$$

The coefficients of A, B, C on the right hand side add up to one, so the right hand side gives the barycentric coordinates of F with respect to A, B, C. Since p is positive the coefficient of C is also positive, and therefore it follows that F and C lie on the same side of AB.

By construction, we have $[AE = [AB$, and therefore $\angle FAE = \angle FAB$. To complete the existence portion of the verification, we need to show that $|\angle FAB| = \theta$ or equivalently $\cos|\angle FAB| = \cos\theta$. By definition, the left hand side is equal to

$$\frac{(F-A)\cdot(B-A)}{|F-A|\cdot|B-A|} \;=\; \frac{\mathbf{y}\cdot|B-A|\mathbf{u}}{|B-A|} \;=\; \mathbf{y}\cdot\mathbf{u}$$

and the by definition the right hand side is equal to $\cos\theta$, which is what we wanted to prove.

We shall now prove the uniqueness part of the statement. Suppose that we have a point D which is on the same side of AB as C and satisfies $|\angle DAB| = \theta$. Write D as a linear combination of A, B, C using barycentric coordinates, so that $D = kA + mB + nC$, where $k + m + n = 1$; since D and C are on the same side of AB, we also have $n > 0$. Then we also have

$$D \;-\; A \;=\; m(B\;-\;A) \;+\; n(C\;-\;A)$$

and since $\mathbf{u}$ and $\mathbf{w}$ form an orthonormal basis for the subspace spanned by $B - A$ and $C - A$ we can also express the right hand side in the form $k\mathbf{u} + \ell\mathbf{w}$ for suitable scalars k and ℓ.

We claim that $\ell > 0$; if we express $\mathbf{u}$ and $\mathbf{w}$ in terms of $B - A$ and $C - A$, we find that

$$D \;-\; A \;=\; \frac{k + q\ell}{|B-A|}\,(B\;-\;A) \;+\; \frac{p\ell}{|C-A|}\,(C\;-\;A)$$

and since p and n are positive it follows that ℓ is also positive.

Finally, suppose that $|\angle DAB| = \theta$; set $r = \sqrt{k^2 + \ell^2}$ (which is just $|D - A|$), and write $\alpha = k/r$, $\beta = \ell/r$; it follows that $r > 0$ and $\beta > 0$. We then have

$$\cos\theta \;=\; \frac{(D-A)\cdot(B-A)}{|D-A|\cdot|B-A|} \;=\; \frac{(r\alpha\,\mathbf{u} \;+\; r\beta\,\mathbf{w})\cdot|B-A|\mathbf{u}}{r\cdot|B-A|} \;=\; \mathbf{y}\cdot\mathbf{u} \;=\; \alpha$$

and therefore the equations $\alpha^2 + \beta^2 = 1$ and $\beta > 0$ imply that $\beta = \sin\theta$. Comparing this with the existence portion of the verification, we see that $D - A = r(F - A)$. It follows that $[AD = [AF$, and therefore we have shown the uniqueness part of the Protractor Postulate.■

Before verifying the Additivity Property for angle measurement, we shall state and prove a fact which will be needed.

LEMMA. *Let W be a plane in $\mathbb{R}^n$, let P be the plane $\mathbf{z} + W$ for some $\mathbf{z} \in \mathbb{R}^n$, let $\mathbf{u}$ and $\mathbf{v}$ form an orthonormal basis for W, and let $L \subset \mathbf{z} + W$ be the line containing $\mathbf{z}$ and $\mathbf{z} + \mathbf{u}$. Let $\mathbf{p} \in P$ be arbitrary, and let $\mathbf{q} = \mathbf{p} - \mathbf{z}$. Then the following hold:*

(i) The vector $\mathbf{p}$ lies on L if and only if $\mathbf{q} \cdot \mathbf{v} = 0$.

(ii) The vector $\mathbf{p}$ lies on the same side of L as $\mathbf{v}$ if and only if $\mathbf{q} \cdot \mathbf{v} > 0$.

(iii) The vector $\mathbf{p}$ lies on the opposite side of L as $\mathbf{v}$ if and only if $\mathbf{z} \cdot \mathbf{v} < 0$.

Proof. By our hypotheses we know that the points

$$\mathbf{z}\ , \quad \mathbf{z} + \mathbf{u}\ , \quad \mathbf{z} + \mathbf{v} \ \in \ P$$

are noncollinear, and if we take the expansion of $\mathbf{p}$ in terms of barycentric coordinates

$$\mathbf{p} \ = \ r\,\mathbf{z} \ + \ s\,(\mathbf{z} + \mathbf{u}) + t\,(\mathbf{z} + \mathbf{w})$$

then the three conditions in the conclusion are equivalent to $t = 0$, $t > 0$ and $t < 0$ respectively. Therefore the result will follow if we can show that $t = \mathbf{q} \cdot \mathbf{v}$. Since the conditions in the lemma imply that $\mathbf{q} = s\,\mathbf{u} + t\,\mathbf{v}$, the equation in the preceding sentence follows immediately (and so does the conclusion of the lemma).■

VERIFICATION OF **(AM–3).** We are given $D \in \mathrm{Int}\,\angle ABC$. If $\theta = |\angle ABD|$ and $\varphi = |\angle DBC|$, then it will suffice to prove the following two statements:

$$\theta + \varphi < \pi \quad \text{and} \quad \cos|\angle ABC| = \cos(\theta + \phi)$$

Since $\cos x$ is strictly decreasing on the interval $[0, \pi]$, the preceding equations imply the additivity identity $|\angle ABC| = \theta + \varphi$.

Define unit vectors $\mathbf{u}$, $\mathbf{x}$ and $\mathbf{y}$ in the plane of $\angle ABC$ as follows:

$$\mathbf{u} \ = \ \frac{1}{|D - B|}\,(D - B)\ , \quad \mathbf{x} \ = \ \frac{1}{|A - B|}\,(A - B)\ , \quad \mathbf{y} \ = \ \frac{1}{|C - B|}\,(C - B)$$

It follows immediately that if $E = B + \mathbf{x}$, $G = B + \mathbf{u}$ and $F = B + \mathbf{y}$, then $[BA = [BE$, $[BD = [BG$ and $[BC = [BF$, so that we have $\theta = |\angle EBG|$ and $\varphi = |\angle GBF|$ and the second statement to be shown is equivalent to $\cos|\angle EBF| = \cos(\theta + \phi)$.

Let W be the 2-dimensional vector subspace of $\mathbb{R}^n$ which is equal or parallel to the plane ABC (note that W must be $\mathbb{R}^2$ if $n = 2$), so that the previously defined unit vectors $\mathbf{u}$, $\mathbf{x}$ and $\mathbf{y}$ all lie in W. By our hypotheses the points E and G lie on the same side of the line $BF = BC$; if $\mathbf{v}_0$ is a unit vector in W which is perpendicular to $\mathbf{u}$, then exactly one of the points $B \pm \mathbf{v}_0$ lies on the same side of BF as A and G; we shall let $\mathbf{v}$ be the corresponding vector of the form $\pm\,\mathbf{v}_0$, and we shall let $K = B + \mathbf{v}$

The Crossbar Theorem implies that E and F lie on opposite sides of BG. Therefore if we take the modified barycentric coordinate expressions

$$E - B \ = \mathbf{x} \ = \ a(G - B) \ + \ b(K - B)\ , \quad F - B \ = \ \mathbf{y} \ = \ c(G - B) \ + \ d(K - B)$$

we see that *the coefficients b and d have* opposite *signs* because one of E, F lies on the same side of BG as K and the other does not. Switching the roles of the variables $\{A, C\}$ and $\{E, F\}$ if necessary, we can assume that $b > 0 > d$. Since the vectors

$$E - B\ , \quad F - B\ , \quad G - B\ , \quad K - B$$

are all unit vectors, it follows that we have

$$a^2 + b^2 = c^2 + d^2 = 1$$

and also that $(a, b) = (\cos\theta, \sin\theta)$ while $(c, d) = (\cos\varphi, -\sin\varphi)$.

Using the definitions and formulas above, we see that

$$\cos|\angle EBF| = ad + bd = \cos(\theta + \varphi)\ .$$

Therefore, if $\theta + \varphi < \pi$ then we must have $|\angle EBF| = \theta + \varphi$, and as noted above this is equivalent to the Additivity Postulate. To conclude the argument, we must show that $\theta + \phi < \pi$.

Suppose, however, that $\theta + \phi \geq \pi$. Since $0 < \theta, \varphi < \pi$, it follows that $\theta + \varphi < 2\pi$, and if we combine these we can conclude that $\sin(\theta + \varphi) \leq 0$. In terms of the coefficients in the preceding paragraphs, this is equivalent to the inequality $bc - ad \leq 0$.

Consider the linear transformation J on W which sends $\mathbf{u}$ to $\mathbf{v}$ and $\mathbf{v}$ into $-\mathbf{u}$; geometrically, J is a 90° rotation, and in any case it is straightforward to verify that J is an orthogonal transformation satisfying $J^2 = -\mathrm{Id}_W$. Since $E = B + \mathbf{x}$ and $G = B + \mathbf{u}$ lie on the same side of BF and $F = B + \mathbf{y}$, by the lemma preceding the start of this verification we know that the signs of the dot products

$$\mathbf{u} \cdot J(\mathbf{y})\ , \mathbf{x} \cdot J(\mathbf{y})$$

are the same. Direct computation shows that

$$J(\mathbf{y}) = c\mathbf{v} - d\mathbf{u}$$

and hence the displayed dot products are equal to $-d$ and $bc - ad$ respectively. We know that d is negative, and therefore both $-d$ and $bc - ad$ are positive. Since $\theta + \varphi \geq \pi$ implies that the second of these is not positive, it follows that we must have $\theta + \varphi < \pi$, and as noted above this completes the verification of the additivity property for angle measurement.■

Congruence Axioms

As noted before, the remarks at the beginning of the previous subheading indicate which types of theorems we are allowed to use; the main point is that we cannot use results which are proved using the Congruence Axioms explicitly. The results in the file

`http://math.ucr.edu/~res/math133/trianglecongruence.pdf`

give purely synthetic proofs that **SAS** implies **ASA** and **SSS**, and accordingly it will suffice to verify **SAS**.

Suppose that we are given ΔABC and ΔDEF such that $d(A, B) = d(D, E)$, $|\angle ABC| = |\angle DEF|$, and $d(B, C) = d(E, F)$. First of all, we can apply the Law of Cosines (Theorem II.3.8 on page 97), whose proof only uses vector geometry, to conclude that $d(A, C) = d(D, F)$. The next step is to prove that and $|\angle ACB| = |\angle DFE|$, and we can also do this using the Law of Cosines as follows: The latter implies that

$$\cos|\angle ACB| = \frac{d(A,C)^2 + d(B,C)^2 - d(A,B)^2}{2\cdot d(A,C)\cdot d(B,C)}$$

$$\cos|\angle DFE| = \frac{d(D,F)^2 + d(E,F)^2 - d(D,E)^2}{2\cdot d(D,F)\cdot d(E,F)}$$

and since we know that the corresponding sides of the triangles have equal lengths it follows that $\cos|\angle ACB| = \cos|\angle DFE|$. If we take inverse cosines of each side, we find that $|\angle ACB| = |\angle DFE|$. Similarly, if we switch the roles of $\{A, C\}$ and $\{D, F\}$ in this argument we conclude that $|\angle CAB| = |\angle FDE|$.■

AXIOM VERIFICATIONS FOR HYPERBOLIC GEOMETRY

One can also verify that the so-called Poincaré unit disk model of hyperbolic geometry (which Beltrami had discovered previously) satisfies all the preceding axioms except for the Parallel Postulate and also verify that the latter is false in that model. However, this requires even more input than the Euclidean case; one reference for the details is Chapter 25 of the book by Moïse.

As noted at the beginning, this verification of the synthetic axioms relies strongly on the following principle:

> *It is possible to define the sine and cosine function and to derive many of their important properties using the methods of calculus without explicitly using anything from geometry.*

For the sake of completeness, we shall summarize how this can be done. Further information appears in Appendix F of the book by Ryan and on pages 182–184 of the following classic text:

> W. Rudin, *Principles of Mathematical Analysis* (Third Ed.). McGraw-Hill, New York *etc.*, 1976.

The basic idea is to start with the standard infinite series for the cosine function and to show that one can derive all the key properties of trigonometric functions from this description by the methods of ordinary single value calculus and the standard theory of second order homogeneous linear equations with constant coefficients. In order to emphasize the avoidance of geometrical methods, we shall denote the standard infinite series for the sine and cosine functions by $\mathsf{SinSeries}\, x$ and $\mathsf{CosSeries}\, x$.

Following the preceding discussion, we make the following definitions:

$$\mathsf{SinSeries}\, x \quad = \quad \sum_{k=0}^{\infty} (-1)^k \, \frac{x^{2k+1}}{(2k+1)!} \ , \qquad \mathsf{CosSeries}\, x \quad = \quad \sum_{k=0}^{\infty} (-1)^k \, \frac{x^{2k}}{(2k)!}$$

The standard Ratio Test for convergence shows that both of these infinite series converge absolutely for all x. Furthermore, since term by term differentiation is valid for convergent power series, it follows that we have the usual differentiation rules:

$$\frac{d}{dx}\, \mathsf{SinSeries}\, x \quad = \quad \mathsf{CosSeries}\, x \ , \qquad \frac{d}{dx}\, \mathsf{CosSeries}\, x \quad = \quad -\mathsf{SinSeries}\, x$$

Combining these, we see that $\mathsf{SinSeries}\, x$ and $\mathsf{CosSeries}\, x$ both solve the second order linear differential equation $y'' = -y$. Furthermore, by standard results on linear differential equations we know that $\mathsf{CosSeries}\, x$ is the unique solution of this differential equation such that $y(0) = 1$ and $y'(0) = 0$, while $\mathsf{SinSeries}\, x$ is the unique solution of this differential equation such that $y(0) = 0$ and $y'(0) = 1$ (most textbooks covering differential equations contain at least a statement of this result, and many contain proofs of the existence and uniqueness theorems need to show uniqueness; some linear algebra textbooks also contain these results).

Since the infinite series for $\mathsf{SinSeries}\, x$ only involves odd powers of x and the infinite series for $\mathsf{CosSeries}\, x$ only involves even powers of x, it follows immediately that these functions are odd and even respectively:

$$\mathsf{SinSeries}\,(-x) \quad = \quad -\,\mathsf{SinSeries}\, x \ , \qquad \mathsf{CosSeries}\,(-x) \quad = \quad \mathsf{CosSeries}\, x$$

Using the material developed thus far we can also establish the following crucial identity :

PROPOSITION. *For all real values of x we have* $\mathsf{SinSeries}^2 x \; + \; \mathsf{CosSeries}^2 x \quad = \quad 1.$

Proof. We know the identity holds when $x = 0$, and if we can show that the left hand side is constant then we may conclude that the identity holds for all x. Proving the left hand side is constant is equivalent to showing that its derivative is zero. But if we differentiate the left hand side we obtain the following:

$$\frac{d}{dx}\left(\mathsf{SinSeries}^2 x \; + \; \mathsf{CosSeries}^2 x\right) \quad =$$

$$2\,(\mathsf{SinSeries}\, x)\,(\mathsf{CosSeries}\, x) \; + \; 2\,(\mathsf{CosSeries}\, x)\,(-\mathsf{SinSeries}\, x) \quad = \quad 0$$

Since the derivative is zero, we know that $\mathsf{SinSeries}^2 x \; + \; \mathsf{CosSeries}^2 x$ is constant and hence must always be equal to 1.

In fact, the same argument also works for all *complex* values of x (if we use the corresponding results from the theory of functions of a complex variable).■

The preceding identity has far-reaching implications, and in particular it leads to results on the oscillating nature of $\mathsf{SinSeries}\, x$ and $\mathsf{CosSeries}\, x$. Perhaps the simplest is that the absolute values of both functions satisfy

$$|\mathsf{SinSeries}\, x| \; , \quad |\mathsf{CosSeries}\, x| \quad \leq \quad 1$$

for all x.

Here are some further consequences of the definition and the proposition; details appear in the citations from Rudin's book.

> *There is at least one positive number x such that* $\mathsf{CosSeries}\, x = 0$. (See the discussion on the last seven lines of page 182).
>
> *If we define π to be the smallest positive real number such that* $\mathsf{CosSeries}(\pi/2) = 0$*, then* $\mathsf{SinSeries}(\pi/2) = 1$. *(See the discussion at the top of page 183.)*
>
> *The functions* $\mathsf{SinSeries}\, x$ *and* $\mathsf{CosSeries}\, x$ *satisfy the usual sum formulas (see the discussion below), and we have* $\mathsf{CosSeries}\, \pi = -1$, $\mathsf{SinSeries}\, \pi = 0$, $\mathsf{CosSeries}\, 2\pi = 1$, $\mathsf{SinSeries}\, 2\pi = 0$. *Furthermore, for all x we also have* $\mathsf{SinSeries}\,(x + 2\pi) = \mathsf{SinSeries}\, x$ *and* $\mathsf{CosSeries}\,(x + 2\pi) = \mathsf{CosSeries}\, x$. (See the subheading "Sum Formulas" below and the discussion on page 183 of Rudin preceding the statement of Theorem 8.7.)

Several other basic properties of the functions $\mathsf{SinSeries}\, x$ and $\mathsf{CosSeries}\, x$ are developed in Theorem 8.7, which is established on pages 183–184 of Rudin.

Remark. The preceding material may seem at least a little pedantic, so we should note similar ideas have important applications to studying the solutions to more general *Sturm-Liouville differential equations* of the form

$$\frac{d}{dx}\left(P(x)\,\frac{dy}{dx}\right) \; + \; \Big(\lambda R(x) - Q(x)\Big)\, y \quad = \quad 0$$

which arise in the study of vibrating objects (in the special case of simple harmonic motion, one has $P(x) = R(x) = 1$ and $Q(x) = 0$, and λ is generally taken to be an integral multiple of some fixed positive number ω_0, which corresponds to the physical notion of fundamental frequency for a vibrating system).

SUM FORMULAS. The theory of differential equations provides a simple way to verify the standard formulas for the sine and cosine of $x \pm y$ in terms of the corresponding functions for x and y:

$$\mathsf{SinSeries}\,(x \pm y) = (\mathsf{SinSeries}\,x)\,(\mathsf{CosSeries}\,y) \pm (\mathsf{CosSeries}\,x)\,(\mathsf{SinSeries}\,y)$$

$$\mathsf{CosSeries}\,(x \pm y) = (\mathsf{CosSeries}\,x)\,(\mathsf{CosSeries}\,y\,) \mp (\mathsf{SinSeries}\,x)\,(\mathsf{SinSeries}\,y)$$

These may be seen as follows: Every solution g to the differential equation $g'' + f = 0$ has the form

$$g(x) = g(0)\,\mathsf{SinSeries}\,x + g'(0)\,\mathsf{CosSeries}\,x$$

by the standard results on homogeneous linear second order differential equations with constant coefficients. In particular, this is true for both $\mathsf{SinSeries}\,(x \pm y)$ and $\mathsf{CosSeries}\,(x \pm y)$ if we view these expressions as functions of x with y held constant. The sum formulas follow by substituting $g(x) = \mathsf{SinSeries}\,(x \pm y)$ and $g(x) = \mathsf{CosSeries}\,(x \pm y)$ into the general formula displayed above.

One can also derive the sum formulas using complex power series together with the standard identities

$$\exp(ix) = \mathsf{CosSeries}\,x + i\,\mathsf{SinSeries}\,x\ , \quad \text{for all} \quad x \in \mathbb{R}$$

$$\exp(z+w) = \exp(z)\cdot\exp(w)\ , \quad \text{for all} \quad z,\ w \in \mathbb{C}\ .$$

See pages 178–179 if Rudin for a detailed discussion.

Final remark. The verifications of the Angle Measurement Axioms also use a simple consequence of the sum formulas; namely, $\mathsf{CosSeries}\,(\pi - x) = -\mathsf{CosSeries}\,x$. One can derive this as follows:

$$\mathsf{CosSeries}\,(\pi - x) = (\mathsf{CosSeries}\,\pi)\,(\mathsf{CosSeries}\,x) + (\mathsf{SinSeries}\,\pi)\,(\mathsf{SinSeries}\,x) =$$

$$(-1)\,\mathsf{CosSeries}\,x + 0\,\mathsf{SinSeries}\,x = -\mathsf{CosSeries}\,x$$

Since we have shown that the power series definitions of the trigonometric functions yield all the properties that are needed to verify the Angle Measurement and Congruence Axioms, in the main body of this document we simply denote these functions by $\sin x$ and $\cos x$ respectively.