Examples

Linear systems of equations $\underset{m \times n}{A}\underset{n \times 1}{X} = \underset{m \times 1}{B}$

Write general solution in the form
$\vec{p} + \mathbb{V}$, $\vec{p}$ = particular solution
$\mathbb{V}$ = solutions to reduced eqn
$AX = 0$. (Find a basis for $\mathbb{V}$)

1. $2x + 3y = 6$ defines line in $\mathbb{R}^2$

Matrix for system:

$$(2 \quad 3 : 6)$$

Row Reduced Echelon form

$$(1 \quad \tfrac{3}{2} : 3) \qquad x + \tfrac{3}{2}y = 3$$

leading nonzero entry

y varies independently, $x = 3 - \frac{3}{2}y$.

Find $\vec{p}$ Set $y = 0$ $\vec{p} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$.

(NOT THE ONLY POSSIBLE CHOICE!!)

Find V Solve $AX=0$. Replace last column in prev. by 0, take echelon form $\left(1 \quad \frac{3}{2} : 0\right)$ $\quad x+\frac{3}{2}y=0$

$x=-\frac{3}{2}y$.

Basic solution (AX=0) Set $y=1$ solve for x.

Basis for V is given by $\begin{pmatrix}-\frac{3}{2}\\1\end{pmatrix}$, so

V = all vectors $t\cdot\begin{pmatrix}-\frac{3}{2}\\1\end{pmatrix}$, where $t\in\mathbb{R}$.

General soln. $AX=B$ $\quad \overset{\vec{p}}{\begin{pmatrix}3\\0\end{pmatrix}}+t\overset{V}{\begin{pmatrix}-\frac{3}{2}\\1\end{pmatrix}}$ $t\in\mathbb{R}$.

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2. $x+y+z=1$ defines plane in $\mathbb{R}^3$

$(A:B)$ is $\left(\underset{\uparrow}{1} \quad 1 \quad 1 : 1\right)$

leading entry variable

y & z vary independently, $x=1-y-z$

Find $\vec{p}$ Set $y=z=0$. $\vec{p}=\begin{pmatrix}1\\0\\0\end{pmatrix}$

Find V Solve $AX=0$. $\left(1 \;\; 1 \;\; 1 : 0\right)$

$x=-y-z$. Basic solutions $\begin{cases} y=0, & z=1 \\ y=1, & z=0. \end{cases}$

The general solution to $AX=0$ is

$$u\begin{pmatrix}-1\\0\\1\end{pmatrix}+v\begin{pmatrix}-1\\1\\0\end{pmatrix} \qquad u,v\in\mathbb{R}$$

$\underline{V}$ = all vectors

General solution to $AX=B$

$$\underbrace{\begin{pmatrix}1\\0\\0\end{pmatrix}}_{\vec{p}}+\underbrace{u\begin{pmatrix}-1\\0\\1\end{pmatrix}+v\begin{pmatrix}-1\\1\\0\end{pmatrix}}_{\underline{V}} \qquad u,v\in\mathbb{R}$$

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3. $\left.\begin{aligned} x+y+z&=4 \\ x-2y+z&=3 \end{aligned}\right\}$ define a line in $\mathbb{R}^3$

$$(A:B)=\begin{pmatrix}1 & 1 & 1 & : 4\\ 1 & -2 & 1 & : 3\end{pmatrix}$$

Put into row reduced echelon form.

(Gauss-Jordan elimination)

$$\left(\begin{array}{ccc:c} 1 & 1 & 1 & 4 \\ 1 & -2 & 1 & 3 \end{array}\right) \xrightarrow[\text{1st}]{\text{2nd minus}} \left(\begin{array}{ccc:c} 1 & 1 & 1 & 4 \\ 0 & -3 & 0 & -1 \end{array}\right)$$

$$\xrightarrow[\text{2nd}]{\frac{1}{3}\times} \left(\begin{array}{ccc:c} 1 & 1 & 1 & 4 \\ 0 & 1 & 0 & \frac{1}{3} \end{array}\right) \xrightarrow[\text{minus 2nd}]{\text{1st}} \left(\begin{array}{ccc:c} 1 & 0 & 1 & 11/3 \\ 0 & 1 & 0 & 1/3 \end{array}\right)$$

leading entry columns

z varies independently, x & y depend on z.

$x = \frac{11}{3} - z \qquad y = \frac{1}{3}$ (indep. of z!)

Find $\vec{p}$ Set $z = 0$ $\quad \vec{p} = \begin{pmatrix} \frac{11}{3} \\ \frac{1}{3} \\ 0 \end{pmatrix}$.

Find $\underline{V}$ Solve system $(A : 0)$ $\begin{cases} x = -z \\ y = 0 \end{cases}$

General solution to $AX = 0$ is $t\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ $t \in \mathbb{R}$.

General solution to $AX = B$ is

$$\underbrace{\begin{pmatrix} \frac{11}{3} \\ \frac{1}{3} \\ 0 \end{pmatrix}}_{\vec{p}} + t \underbrace{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}}_{\underline{V}} \qquad t \in \mathbb{R}.$$

4. Same type as #3:

$$\left.\begin{aligned} x + 2y + z &= 6 \\ x - 2y + z &= 10 \end{aligned}\right\}$$ interse equation defines a line in $\mathbb{R}^3$

$$(A:B) = \left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 1 & -2 & 1 & 10 \end{array}\right)$$ Row reduce:

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 1 & -2 & 1 & 10 \end{array}\right) \xrightarrow[\text{1st}]{\text{2nd minus}} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -4 & -2 & 4 \end{array}\right) \xrightarrow[\text{2nd}]{-\frac{1}{4}\text{ times}}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & 1 & \frac{1}{2} & -1 \end{array}\right) \xrightarrow[2 \times \text{2nd}]{\text{1st minus}} \left(\begin{array}{ccc|c} 1 & 0 & 2 & 8 \\ 0 & 1 & \frac{1}{2} & -1 \end{array}\right)$$

↑ ↑ leading columns.

z arbitrary, x & y depend on z.

Find $\vec{p}$: Set $z = 0$, get $\vec{p} = \begin{pmatrix} 8 \\ -1 \\ 0 \end{pmatrix}$

Find basis for V: Reduced system $\left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & \frac{1}{2} & 0 \end{array}\right)$, basis given by $z = 1$, so a basis vector is

$$\begin{pmatrix} -2 \\ -\frac{1}{2} \\ 1 \end{pmatrix}.$$

General solution is

$$\begin{pmatrix} 8 \\ -1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -2 \\ -\frac{1}{2} \\ 1 \end{pmatrix}, \quad t \in \mathbb{R}$$