7. *Hint:* If $A = R^T R$, where R is invertible, then A is positive definite, by Exercise 25 in Section 7.2. Conversely, suppose that A is positive definite. Then by Exercise 26 in Section 7.2, $A = B^T B$ for some positive definite matrix B. Explain why B admits a QR factorization, and use it to create the Cholesky factorization of A.

9. If A is $m \times n$ and $\mathbf{x}$ is in $\mathbb{R}^n$, then $\mathbf{x}^T A^T A\mathbf{x} = (A\mathbf{x})^T(A\mathbf{x}) = \|A\mathbf{x}\|^2 \geq 0$. Thus $A^T A$ is positive semidefinite. By Exercise 22 in Section 6.5, rank $A^T A$ = rank A.

11. *Hint:* Write an SVD of A in the form $A = U\Sigma V^T = PQ$, where $P = U\Sigma U^T$ and $Q = UV^T$. Show that P is symmetric and has the same eigenvalues as Σ. Explain why Q is an orthogonal matrix.

13. **a.** If $\mathbf{b} = A\mathbf{x}$, then $\mathbf{x}^+ = A^+\mathbf{b} = A^+A\mathbf{x}$. By Exercise 12(a), $\mathbf{x}^+$ is the orthogonal projection of $\mathbf{x}$ onto Row A.

b. From (a) and then Exercise 12(c), $A\mathbf{x}^+ = A(A^+A\mathbf{x}) = (AA^+A)\mathbf{x} = A\mathbf{x} = \mathbf{b}$.

c. Since $\mathbf{x}^+$ is the orthogonal projection onto Row A, the Pythagorean Theorem shows that $\|\mathbf{u}\|^2 = \|\mathbf{x}^+\|^2 + \|\mathbf{u} - \mathbf{x}^+\|^2$. Part (c) follows immediately.

15. **[M]** $A^+ = \frac{1}{40} \cdot \begin{bmatrix} -2 & -14 & 13 & 13 \\ -2 & -14 & 13 & 13 \\ -2 & 6 & -7 & -7 \\ 2 & -6 & 7 & 7 \\ 4 & -12 & -6 & -6 \end{bmatrix}$, $\hat{\mathbf{x}} = \begin{bmatrix} .7 \\ .7 \\ -.8 \\ .8 \\ .6 \end{bmatrix}$

The reduced echelon form of $\begin{bmatrix} A \\ \mathbf{x}^T \end{bmatrix}$ is the same as the reduced echelon form of A, except for an extra row of zeros. So adding scalar multiples of the rows of A to $\mathbf{x}^T$ can produce the zero vector, which shows that $\mathbf{x}^T$ is in Row A.

Basis for Nul A: $\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$

Chapter 8

Section 8.1, page 444

1. Some possible answers: $\mathbf{y} = 2\mathbf{v}_1 - 1.5\mathbf{v}_2 + .5\mathbf{v}_3$, $\mathbf{y} = 2\mathbf{v}_1 - 2\mathbf{v}_3 + \mathbf{v}_4$, $\mathbf{y} = 2\mathbf{v}_1 + 3\mathbf{v}_2 - 7\mathbf{v}_3 + 3\mathbf{v}_4$

3. $\mathbf{y} = -3\mathbf{v}_1 + 2\mathbf{v}_2 + 2\mathbf{v}_3$. The weights sum to 1, so this is an affine sum.

5. **a.** $\mathbf{p}_1 = 3\mathbf{b}_1 - \mathbf{b}_2 - \mathbf{b}_3 \in \text{aff } S$ since the coefficients sum to 1.

b. $\mathbf{p}_2 = 2\mathbf{b}_1 + 0\mathbf{b}_2 + \mathbf{b}_3 \notin \text{aff } S$ since the coefficients do not sum to 1.

c. $\mathbf{p}_3 = -\mathbf{b}_1 + 2\mathbf{b}_2 + 0\mathbf{b}_3 \in \text{aff } S$ since the coefficients sum to 1.

7. **a.** $\mathbf{p}_1 \in \text{Span } S$, but $\mathbf{p}_1 \notin \text{aff } S$

b. $\mathbf{p}_2 \in \text{Span } S$, and $\mathbf{p}_2 \in \text{aff } S$

c. $\mathbf{p}_3 \notin \text{Span } S$, so $\mathbf{p}_3 \notin \text{aff } S$

9. $\mathbf{v}_1 = \begin{bmatrix} -3 \\ 0 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$. Other answers are possible.

11. See the *Study Guide*.

13. Span $\{\mathbf{v}_2 - \mathbf{v}_1, \mathbf{v}_3 - \mathbf{v}_1\}$ is a plane if and only if $\{\mathbf{v}_2 - \mathbf{v}_1, \mathbf{v}_3 - \mathbf{v}_1\}$ is linearly independent. Suppose c_2 and c_3 satisfy $c_2(\mathbf{v}_2 - \mathbf{v}_1) + c_3(\mathbf{v}_3 - \mathbf{v}_1) = \mathbf{0}$. Show that this implies $c_2 = c_3 = 0$.

15. Let $S = \{\mathbf{x} : A\mathbf{x} = \mathbf{b}\}$. To show that S is affine, it suffices to show that S is a flat, by Theorem 3. Let $W = \{\mathbf{x} : A\mathbf{x} = \mathbf{0}\}$. Then W is a subspace of $\mathbb{R}^n$, by Theorem 2 in Section 4.2 (or Theorem 12 in Section 2.8). Since $S = W + \mathbf{p}$, where $\mathbf{p}$ satisfies $A\mathbf{p} = \mathbf{b}$, by Theorem 6 in Section 1.5, S is a translate of W, and hence S is a flat.

17. A suitable set consists of any three vectors that are not collinear and have 5 as their third entry. If 5 is their third entry, they lie in the plane $z = 5$. If the vectors are not collinear, their affine hull cannot be a line, so it must be the plane.

19. If $\mathbf{p}, \mathbf{q} \in f(S)$, then there exist $\mathbf{r}, \mathbf{s} \in S$ such that $f(\mathbf{r}) = \mathbf{p}$ and $f(\mathbf{s}) = \mathbf{q}$. Given any $t \in \mathbb{R}$, we must show that $\mathbf{z} = (1 - t)\mathbf{p} + t\mathbf{q}$ is in $f(S)$. Now use definitions of $\mathbf{p}$ and $\mathbf{q}$, and the fact that f is linear. The complete proof is presented in the *Study Guide*.

21. Since B is affine, Theorem 2 implies that B contains all affine combinations of points of B. Hence B contains all affine combinations of points of A. That is, aff $A \subset B$.

23. Since $A \subset (A \cup B)$, it follows from Exercise 22 that aff $A \subset$ aff $(A \cup B)$. Similarly, aff $B \subset$ aff $(A \cup B)$, so [aff $A \cup$ aff B] $\subset$ aff $(A \cup B)$.

25. To show that $D \subset E \cap F$, show that $D \subset E$ and $D \subset F$. The complete proof is presented in the *Study Guide*.

Section 8.2, page 454

1. Affinely dependent and $2\mathbf{v}_1 + \mathbf{v}_2 - 3\mathbf{v}_3 = \mathbf{0}$

3. The set is affinely independent. If the points are called $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$, then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for $\mathbb{R}^3$ and $\mathbf{v}_4 = 16\mathbf{v}_1 + 5\mathbf{v}_2 - 3\mathbf{v}_3$, but the weights in the linear combination do not sum to 1.

5. $-4\mathbf{v}_1 + 5\mathbf{v}_2 - 4\mathbf{v}_3 + 3\mathbf{v}_4 = \mathbf{0}$

7. The barycentric coordinates are $(-2, 4, -1)$.

9. See the *Study Guide*.

11. When a set of five points is translated by subtracting, say, the first point, the new set of four points must be linearly dependent, by Theorem 8 in Section 1.7, because the four points are in $\mathbb{R}^3$. By Theorem 5, the original set of five points is affinely dependent.

13. If $\{\mathbf{v}_1, \mathbf{v}_2\}$ is affinely dependent, then there exist c_1 and c_2, not both zero, such that $c_1 + c_2 = 0$ and $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{0}$. Show that this implies $\mathbf{v}_1 = \mathbf{v}_2$. For the converse, suppose $\mathbf{v}_1 = \mathbf{v}_2$ and select specific c_1 and c_2 that show their affine dependence. The details are in the *Study Guide*.

15. **a.** The vectors $\mathbf{v}_2 - \mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mathbf{v}_3 - \mathbf{v}_1 = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ are not multiples and hence are linearly independent. By Theorem 5, S is affinely independent.

b. $\mathbf{p}_1 \leftrightarrow \left(-\frac{6}{8}, \frac{9}{8}, \frac{5}{8}\right)$, $\mathbf{p}_2 \leftrightarrow \left(0, \frac{1}{2}, \frac{1}{2}\right)$, $\mathbf{p}_3 \leftrightarrow \left(\frac{14}{8}, -\frac{5}{8}, -\frac{1}{8}\right)$, $\mathbf{p}_4 \leftrightarrow \left(\frac{6}{8}, -\frac{5}{8}, \frac{7}{8}\right)$, $\mathbf{p}_5 \leftrightarrow \left(\frac{1}{4}, \frac{1}{8}, \frac{5}{8}\right)$

c. $\mathbf{p}_6$ is $(-, -, +)$, $\mathbf{p}_7$ is $(0, +, -)$, and $\mathbf{p}_8$ is $(+, +, -)$.

17. Suppose $S = \{\mathbf{b}_1, \dots, \mathbf{b}_k\}$ is an affinely independent set. Then equation (7) has a solution, because $\mathbf{p}$ is in aff S. Hence equation (8) has a solution. By Theorem 5, the homogeneous forms of the points in S are linearly independent. Thus (8) has a unique solution. Then (7) also has a unique solution, because (8) encodes both equations that appear in (7).

The following argument mimics the proof of Theorem 7 in Section 4.4. If $S = \{\mathbf{b}_1, \dots, \mathbf{b}_k\}$ is an affinely independent set, then scalars $c_1, \dots, c_k$ exist that satisfy (7), by definition of aff S. Suppose $\mathbf{x}$ also has the representation

$$\mathbf{x} = d_1\mathbf{b}_1 + \cdots + d_k\mathbf{b}_k \quad \text{and} \quad d_1 + \cdots + d_k = 1 \qquad \text{(7a)}$$

for scalars $d_1, \dots, d_k$. Then subtraction produces the equation

$$\mathbf{0} = \mathbf{x} - \mathbf{x} = (c_1 - d_1)\mathbf{b}_1 + \cdots + (c_k - d_k)\mathbf{b}_k \qquad \text{(7b)}$$

The weights in (7b) sum to 0 because the c's and the d's separately sum to 1. This is impossible, unless each weight in (8) is 0, because S is an affinely independent set. This proves that $c_i = d_i$ for $i = 1, \dots, k$.

19. If $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ is an affinely dependent set, then there exist scalars c_1, c_2, and c_3, not all zero, such that $c_1\mathbf{p}_1 + c_2\mathbf{p}_2 + c_3\mathbf{p}_3 = \mathbf{0}$ and $c_1 + c_2 + c_3 = 0$. Now use the linearity of f.

21. Let $\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$, and $\mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$. Then

$\det[\,\tilde{\mathbf{a}} \;\; \tilde{\mathbf{b}} \;\; \tilde{\mathbf{c}}\,] = \det \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ 1 & 1 & 1 \end{bmatrix} =$

$\det \begin{bmatrix} a_1 & a_2 & 1 \\ b_1 & b_2 & 1 \\ c_1 & c_2 & 1 \end{bmatrix}$, by the transpose property of the determinant (Theorem 5 in Section 3.2). By Exercise 30 in Section 3.3, this determinant equals 2 times the area of the triangle with vertices at $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$.

23. If $[\,\tilde{\mathbf{a}} \;\; \tilde{\mathbf{b}} \;\; \tilde{\mathbf{c}}\,]\begin{bmatrix} r \\ s \\ t \end{bmatrix} = \tilde{\mathbf{p}}$, then Cramer's rule gives

$r = \det[\,\tilde{\mathbf{p}} \;\; \tilde{\mathbf{b}} \;\; \tilde{\mathbf{c}}\,] / \det[\,\tilde{\mathbf{a}} \;\; \tilde{\mathbf{b}} \;\; \tilde{\mathbf{c}}\,]$. By Exercise 21, the numerator of this quotient is twice the area of $\triangle\mathbf{pbc}$, and the denominator is twice the area of $\triangle\mathbf{abc}$. This proves the formula for r. The other formulas are proved using Cramer's rule for s and t.

25. The intersection point is $\mathbf{x}(4) =$

$$-.1\begin{bmatrix} 1 \\ 3 \\ -6 \end{bmatrix} + .6\begin{bmatrix} 7 \\ 3 \\ -5 \end{bmatrix} + .5\begin{bmatrix} 3 \\ 9 \\ -2 \end{bmatrix} = \begin{bmatrix} 5.6 \\ 6.0 \\ -3.4 \end{bmatrix}.$$

It is not inside the triangle.

Section 8.3, page 461

1. See the *Study Guide*.

3. None are in conv S.

5. $\mathbf{p}_1 = -\frac{1}{6}\mathbf{v}_1 + \frac{1}{3}\mathbf{v}_2 + \frac{2}{3}\mathbf{v}_3 + \frac{1}{6}\mathbf{v}_4$, so $\mathbf{p}_1 \notin$ conv S.
$\mathbf{p}_2 = \frac{1}{3}\mathbf{v}_1 + \frac{1}{3}\mathbf{v}_2 + \frac{1}{6}\mathbf{v}_3 + \frac{1}{6}\mathbf{v}_4$, so $\mathbf{p}_2 \in$ conv S.

7. **a.** The barycentric coordinates of $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$, and $\mathbf{p}_4$ are, respectively, $\left(\frac{1}{3}, \frac{1}{6}, \frac{1}{2}\right)$, $\left(0, \frac{1}{2}, \frac{1}{2}\right)$, $\left(\frac{1}{2}, -\frac{1}{4}, \frac{3}{4}\right)$, and $\left(\frac{1}{2}, \frac{3}{4}, -\frac{1}{4}\right)$.

b. $\mathbf{p}_3$ and $\mathbf{p}_4$ are outside conv T. $\mathbf{p}_1$ is inside conv T. $\mathbf{p}_2$ is on the edge $\overline{\mathbf{v}_2\mathbf{v}_3}$ of conv T.

9. $\mathbf{p}_1$ and $\mathbf{p}_3$ are outside the tetrahedron conv S. $\mathbf{p}_2$ is on the face containing the vertices $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$. $\mathbf{p}_4$ is inside conv S. $\mathbf{p}_5$ is on the edge between $\mathbf{v}_1$ and $\mathbf{v}_3$.

11. See the *Study Guide*.

13. If $\mathbf{p}, \mathbf{q} \in f(S)$, then there exist $\mathbf{r}, \mathbf{s} \in S$ such that $f(\mathbf{r}) = \mathbf{p}$ and $f(\mathbf{s}) = \mathbf{q}$. The goal is to show that the line segment $\mathbf{y} = (1 - t)\mathbf{p} + t\mathbf{q}$, for $0 \le t \le 1$, is in $f(S)$. Use the linearity of f and the convexity of S to show that $\mathbf{y} = f(\mathbf{w})$ for some $\mathbf{w}$ in S. This will show that $\mathbf{y}$ is in $f(S)$ and that $f(S)$ is convex.

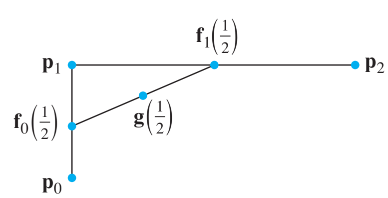
15. $\mathbf{p} = \frac{1}{6}\mathbf{v}_1 + \frac{1}{2}\mathbf{v}_2 + \frac{1}{3}\mathbf{v}_4$ and $\mathbf{p} = \frac{1}{2}\mathbf{v}_1 + \frac{1}{6}\mathbf{v}_2 + \frac{1}{3}\mathbf{v}_3$.

17. Suppose $A \subset B$, where B is convex. Then, since B is convex, Theorem 7 implies that B contains all convex combinations of points of B. Hence B contains all convex combinations of points of A. That is, conv $A \subset B$.

19. **a.** Use Exercise 18 to show that conv A and conv B are both subsets of conv $(A \cup B)$. This will imply that their union is also a subset of conv $(A \cup B)$.

b. One possibility is to let A be two adjacent corners of a square and let B be the other two corners. Then what is (conv A) $\cup$ (conv B), and what is conv $(A \cup B)$?

21.



23.
$$\begin{aligned} \mathbf{g}(t) &= (1 - t)\mathbf{f}_0(t) + t\mathbf{f}_1(t) \\ &= (1 - t)[(1 - t)\mathbf{p}_0 + t\mathbf{p}_1] + t[(1 - t)\mathbf{p}_1 + t\mathbf{p}_2] \\ &= (1 - t)^2\mathbf{p}_0 + 2t(1 - t)\mathbf{p}_1 + t^2\mathbf{p}_2. \end{aligned}$$

The sum of the weights in the linear combination for $\mathbf{g}$ is $(1-t)^2 + 2t(1-t) + t^2$, which equals $(1-2t+t^2) + (2t-2t^2) + t^2 = 1$. The weights are each between 0 and 1 when $0 \le t \le 1$, so $\mathbf{g}(t)$ is in conv $\{\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2\}$.

Section 8.4, page 469

1. $f(x_1, x_2) = 3x_1 + 4x_2$ and $d = 13$

3. **a.** Open **b.** Closed **c.** Neither
d. Closed **e.** Closed

5. **a.** Not compact, convex
b. Compact, convex
c. Not compact, convex
d. Not compact, not convex
e. Not compact, convex

7. **a.** $\mathbf{n} = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$ or a multiple
b. $f(\mathbf{x}) = 2x_2 + 3x_3$, $d = 11$

9. **a.** $\mathbf{n} = \begin{bmatrix} 3 \\ -1 \\ 2 \\ 1 \end{bmatrix}$ or a multiple
b. $f(\mathbf{x}) = 3x_1 - x_2 + 2x_3 + x_4$, $d = 5$

11. $\mathbf{v}_2$ is on the same side as $\mathbf{0}$, $\mathbf{v}_1$ is on the other side, and $\mathbf{v}_3$ is in H.

13. One possibility is $\mathbf{p} = \begin{bmatrix} 32 \\ -14 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{v}_1 = \begin{bmatrix} 10 \\ -7 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -4 \\ 1 \\ 0 \\ 1 \end{bmatrix}$.

15. $f(x_1, x_2, x_3, x_4) = x_1 - 3x_2 + 4x_3 - 2x_4$, and $d = 5$

17. $f(x_1, x_2, x_3) = x_1 - 2x_2 + x_3$, and $d = 0$

19. $f(x_1, x_2, x_3) = -5x_1 + 3x_2 + x_3$, and $d = 0$

21. See the *Study Guide*.

23. $f(x_1, x_2) = 3x_1 - 2x_2$ with d satisfying $9 < d < 10$ is one possibility.

25. $f(x, y) = 4x + y$. A natural choice for d is 12.75, which equals $f(3, .75)$. The point $(3, .75)$ is three-fourths of the distance between the center of $B(\mathbf{0}, 3)$ and the center of $B(\mathbf{p}, 1)$.

27. Exercise 2(a) in Section 8.3 gives one possibility. Or let $S = \{(x, y) : x^2y^2 = 1 \text{ and } y > 0\}$. Then conv S is the upper (open) half-plane.

29. Let $\mathbf{x}, \mathbf{y} \in B(\mathbf{p}, \delta)$ and suppose $\mathbf{z} = (1-t)\mathbf{x} + t\mathbf{y}$, where $0 \le t \le 1$. Then show that

$$\begin{aligned} \|\mathbf{z} - \mathbf{p}\| &= \|[(1-t)\mathbf{x} + t\mathbf{y}] - \mathbf{p}\| \\ &= \|(1-t)(\mathbf{x} - \mathbf{p}) + t(\mathbf{y} - \mathbf{p})\| < \delta. \end{aligned}$$

Section 8.5, page 481

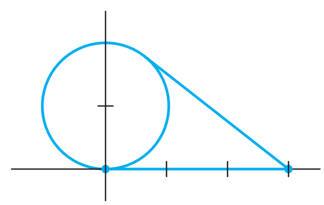
1. **a.** $m = 1$ at the point $\mathbf{p}_1$ **b.** $m = 5$ at the point $\mathbf{p}_2$
c. $m = 5$ at the point $\mathbf{p}_3$

3. **a.** $m = -3$ at the point $\mathbf{p}_3$
b. $m = 1$ on the set conv $\{\mathbf{p}_1, \mathbf{p}_3\}$
c. $m = -3$ on the set conv $\{\mathbf{p}_1, \mathbf{p}_2\}$

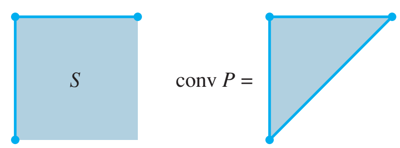
5. $\left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \end{bmatrix} \right\}$

7. $\left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 7 \\ 0 \end{bmatrix}, \begin{bmatrix} 6 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 6 \end{bmatrix} \right\}$

9. The origin is an extreme point, but it is not a vertex. Explain why.



11. One possibility is to let S be a square that includes part of the boundary but not all of it. For example, include just two adjacent edges. The convex hull of the profile P is a triangular region.



13. **a.** $f_0(C^5) = 32$, $f_1(C^5) = 80$, $f_2(C^5) = 80$, $f_3(C^5) = 40$, $f_4(C^5) = 10$, and $32 - 80 + 80 - 40 + 10 = 2$.

b.

	f_0	f_1	f_2	f_3	f_4
C^1	2				
C^2	4	4			
C^3	8	12	6		
C^4	16	32	24	8	
C^5	32	80	80	40	10

For a general formula, see the *Study Guide*.

15. **a.** $f_0(P^n) = f_0(Q) + 1$
b. $f_k(P^n) = f_k(Q) + f_{k-1}(Q)$
c. $f_{n-1}(P^n) = f_{n-2}(Q) + 1$

17. See the *Study Guide*.

19. Let S be convex and let $\mathbf{x} \in cS + dS$, where $c > 0$ and $d > 0$. Then there exist $\mathbf{s}_1$ and $\mathbf{s}_2$ in S such that $\mathbf{x} = c\mathbf{s}_1 + d\mathbf{s}_2$. But then

$$\mathbf{x} = c\mathbf{s}_1 + d\mathbf{s}_2 = (c+d)\left(\frac{c}{c+d}\mathbf{s}_1 + \frac{d}{c+d}\mathbf{s}_2\right).$$

Now show that the expression on the right side is a member of $(c+d)S$.

For the converse, pick a typical point in $(c+d)S$ and show it is in $cS + dS$.

21. *Hint:* Suppose A and B are convex. Let $\mathbf{x}, \mathbf{y} \in A + B$. Then there exist $\mathbf{a}, \mathbf{c} \in A$ and $\mathbf{b}, \mathbf{d} \in B$ such that $\mathbf{x} = \mathbf{a} + \mathbf{b}$ and $\mathbf{y} = \mathbf{c} + \mathbf{d}$. For any t such that $0 \le t \le 1$, show that

$$\mathbf{w} = (1-t)\mathbf{x} + t\mathbf{y} = (1-t)(\mathbf{a}+\mathbf{b}) + t(\mathbf{c}+\mathbf{d})$$

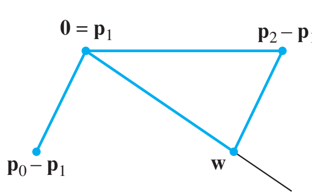
represents a point in $A + B$.

Section 8.6, page 492

1. The control points for $\mathbf{x}(t) + \mathbf{b}$ should be $\mathbf{p}_0 + \mathbf{b}$, $\mathbf{p}_1 + \mathbf{b}$, and $\mathbf{p}_3 + \mathbf{b}$. Write the Bézier curve through these points, and show algebraically that this curve is $\mathbf{x}(t) + \mathbf{b}$. See the *Study Guide*.

3. a. $\mathbf{x}'(t) = (-3 + 6t - 3t^2)\mathbf{p}_0 + (3 - 12t + 9t^2)\mathbf{p}_1 + (6t - 9t^2)\mathbf{p}_2 + 3t^2\mathbf{p}_3$, so
$\mathbf{x}'(0) = -3\mathbf{p}_0 + 3\mathbf{p}_1 = 3(\mathbf{p}_1 - \mathbf{p}_0)$, and
$\mathbf{x}'(1) = -3\mathbf{p}_2 + 3\mathbf{p}_3 = 3(\mathbf{p}_3 - \mathbf{p}_2)$. This shows that the tangent vector $\mathbf{x}'(0)$ points in the direction from $\mathbf{p}_0$ to $\mathbf{p}_1$ and is three times the length of $\mathbf{p}_1 - \mathbf{p}_0$. Likewise, $\mathbf{x}'(1)$ points in the direction from $\mathbf{p}_2$ to $\mathbf{p}_3$ and is three times the length of $\mathbf{p}_3 - \mathbf{p}_2$. In particular, $\mathbf{x}'(1) = \mathbf{0}$ if and only if $\mathbf{p}_3 = \mathbf{p}_2$.

b. $\mathbf{x}''(t) = (6 - 6t)\mathbf{p}_0 + (-12 + 18t)\mathbf{p}_1$
$\quad + (6 - 18t)\mathbf{p}_2 + 6t\mathbf{p}_3$, so that
$\mathbf{x}''(0) = 6\mathbf{p}_0 - 12\mathbf{p}_1 + 6\mathbf{p}_2 = 6(\mathbf{p}_0 - \mathbf{p}_1) + 6(\mathbf{p}_2 - \mathbf{p}_1)$
and
$\mathbf{x}''(1) = 6\mathbf{p}_1 - 12\mathbf{p}_2 + 6\mathbf{p}_3 = 6(\mathbf{p}_1 - \mathbf{p}_2) + 6(\mathbf{p}_3 - \mathbf{p}_2)$
For a picture of $\mathbf{x}''(0)$, construct a coordinate system with the origin at $\mathbf{p}_1$, temporarily, label $\mathbf{p}_0$ as $\mathbf{p}_0 - \mathbf{p}_1$, and label $\mathbf{p}_2$ as $\mathbf{p}_2 - \mathbf{p}_1$. Finally, construct a line from this new origin through the sum of $\mathbf{p}_0 - \mathbf{p}_1$ and $\mathbf{p}_2 - \mathbf{p}_1$, extended out a bit. That line points in the direction of $\mathbf{x}''(0)$.



$\mathbf{w} = (\mathbf{p}_0 - \mathbf{p}_1) + (\mathbf{p}_2 - \mathbf{p}_1) = \frac{1}{6}\mathbf{x}''(0)$

5. a. From Exercise 3(a) or equation (9) in the text,

$$\mathbf{x}'(1) = 3(\mathbf{p}_3 - \mathbf{p}_2)$$

Use the formula for $\mathbf{x}'(0)$, with the control points from $\mathbf{y}(t)$, and obtain

$$\mathbf{y}'(0) = -3\mathbf{p}_3 + 3\mathbf{p}_4 = 3(\mathbf{p}_4 - \mathbf{p}_3)$$

For C^1 continuity, $3(\mathbf{p}_3 - \mathbf{p}_2) = 3(\mathbf{p}_4 - \mathbf{p}_3)$, so $\mathbf{p}_3 = (\mathbf{p}_4 + \mathbf{p}_2)/2$, and $\mathbf{p}_3$ is the midpoint of the line segment from $\mathbf{p}_2$ to $\mathbf{p}_4$.

b. If $\mathbf{x}'(1) = \mathbf{y}'(0) = \mathbf{0}$, then $\mathbf{p}_2 = \mathbf{p}_3$ and $\mathbf{p}_3 = \mathbf{p}_4$. Thus, the "line segment" from $\mathbf{p}_2$ to $\mathbf{p}_4$ is just the point $\mathbf{p}_3$. [*Note:* In this case, the combined curve is still C^1 continuous, by definition. However, some choices of the other "control" points, $\mathbf{p}_0$, $\mathbf{p}_1$, $\mathbf{p}_5$, and $\mathbf{p}_6$, can produce a curve with a visible corner at $\mathbf{p}_3$, in which case the curve is not G^1 continuous at $\mathbf{p}_3$.]

7. *Hint:* Use $\mathbf{x}''(t)$ from Exercise 3 and adapt this for the second curve to see that

$$\mathbf{y}''(t) = 6(1-t)\mathbf{p}_3 + 6(-2+3t)\mathbf{p}_4 + 6(1-3t)\mathbf{p}_5 + 6t\mathbf{p}_6$$

Then set $\mathbf{x}''(1) = \mathbf{y}''(0)$. Since the curve is C^1 continuous at $\mathbf{p}_3$, Exercise 5(a) says that the point $\mathbf{p}_3$ is the midpoint of the segment from $\mathbf{p}_2$ to $\mathbf{p}_4$. This implies that $\mathbf{p}_4 - \mathbf{p}_3 = \mathbf{p}_3 - \mathbf{p}_2$. Use this substitution to show that $\mathbf{p}_4$ and $\mathbf{p}_5$ are uniquely determined by $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$. Only $\mathbf{p}_6$ can be chosen arbitrarily.

9. Write a vector of the polynomial weights for $\mathbf{x}(t)$, expand the polynomial weights, and factor the vector as $M_B\mathbf{u}(t)$:

$$\begin{bmatrix} 1 - 4t + 6t^2 - 4t^3 + t^4 \\ 4t - 12t^2 + 12t^3 - 4t^4 \\ 6t^2 - 12t^3 + 6t^4 \\ 4t^3 - 4t^4 \\ t^4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -4 & 6 & -4 & 1 \\ 0 & 4 & -12 & 12 & -4 \\ 0 & 0 & 6 & -12 & 6 \\ 0 & 0 & 0 & 4 & -4 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ t \\ t^2 \\ t^3 \\ t^4 \end{bmatrix},$$

$$M_B = \begin{bmatrix} 1 & -4 & 6 & -4 & 1 \\ 0 & 4 & -12 & 12 & -4 \\ 0 & 0 & 6 & -12 & 6 \\ 0 & 0 & 0 & 4 & -4 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

11. See the *Study Guide*.

13. a. *Hint:* Use the fact that $\mathbf{q}_0 = \mathbf{p}_0$.

b. Multiply the first and last parts of equation (13) by $\frac{8}{3}$ and solve for $8\mathbf{q}_2$.

c. Use equation (8) to substitute for $8\mathbf{q}_3$ and then apply part (a).

15. a. From equation (11), $\mathbf{y}'(1) = .5\mathbf{x}'(.5) = \mathbf{z}'(0)$.

b. Observe that $\mathbf{y}'(1) = 3(\mathbf{q}_3 - \mathbf{q}_2)$. This follows from equation (9), with $\mathbf{y}(t)$ and its control points in place of $\mathbf{x}(t)$ and its control points. Similarly, for $\mathbf{z}(t)$ and its control points, $\mathbf{z}'(0) = 3(\mathbf{r}_1 - \mathbf{r}_0)$. By part (a),

$3(\mathbf{q}_3 - \mathbf{q}_2) = 3(\mathbf{r}_1 - \mathbf{r}_0)$. Replace $\mathbf{r}_0$ by $\mathbf{q}_3$, and obtain $\mathbf{q}_3 - \mathbf{q}_2 = \mathbf{r}_1 - \mathbf{q}_3$, and hence $\mathbf{q}_3 = (\mathbf{q}_2 + \mathbf{r}_1)/2$.

c. Set $\mathbf{q}_0 = \mathbf{p}_0$ and $\mathbf{r}_3 = \mathbf{p}_3$. Compute $\mathbf{q}_1 = (\mathbf{p}_0 + \mathbf{p}_1)/2$ and $\mathbf{r}_2 = (\mathbf{p}_2 + \mathbf{p}_3)/2$. Compute $\mathbf{m} = (\mathbf{p}_1 + \mathbf{p}_2)/2$. Compute $\mathbf{q}_2 = (\mathbf{q}_1 + \mathbf{m})/2$ and $\mathbf{r}_1 = (\mathbf{m} + \mathbf{r}_2)/2$. Compute $\mathbf{q}_3 = (\mathbf{q}_2 + \mathbf{r}_1)/2$ and set $\mathbf{r}_0 = \mathbf{q}_3$.

17. a. $\mathbf{r}_0 = \mathbf{p}_0, \mathbf{r}_1 = \dfrac{\mathbf{p}_0 + 2\mathbf{p}_1}{3}, \mathbf{r}_2 = \dfrac{2\mathbf{p}_1 + \mathbf{p}_2}{3}, \mathbf{r}_3 = \mathbf{p}_2$

b. *Hint:* Write the standard formula (7) in this section, with $\mathbf{r}_i$ in place of $\mathbf{p}_i$ for $i = 0, \ldots, 3$, and then replace $\mathbf{r}_0$ and $\mathbf{r}_3$ by $\mathbf{p}_0$ and $\mathbf{p}_2$, respectively:

$$\begin{aligned}\mathbf{x}(t) = {} & (1 - 3t + 3t^2 - t^3)\mathbf{p}_0 \\ & + (3t - 6t^2 + 3t^3)\mathbf{r}_1 \\ & + (3t^2 - 3t^3)\mathbf{r}_2 + t^3\mathbf{p}_2\end{aligned} \qquad (iii)$$

Use the formulas for $\mathbf{r}_1$ and $\mathbf{r}_2$ from part (a) to examine the second and third terms in this expression for $\mathbf{x}(t)$.