Further comments on axioms for geometry

One important feature of the ***Elements*** is that it develops geometry from a very short list of assumptions. Although modern axiom systems like Hilbert's (or G. D. Birkhoff's) eliminate the logical deficiencies of Euclid's framework in the ***Elements***, they are certainly far less concise and require several additional undefined concepts. For several reasons it is worthwhile to know whether there are alternatives which require fewer undefined concepts and postulates about these concepts. Here are a few:

1. **Brevity.** The conciseness of Euclid's assumptions is one of the most striking features of his work, and it illustrates the power of logical deduction very convincingly. One would like a system that is logically rigorous but not so long that it dilutes this impression any more than it absolutely necessary.
2. **Consistency.** Whenever one makes a list of assumptions, it is fair to ask for some assurance that they do not lead to logical contradictions. Fewer assumptions allow one to study such issues of logical consistency more effectively.
3. **Clarity.** Notwithstanding the points raised above, for many points it is also desirable to have working versions of axiomatic systems that are not so terse that they are needlessly difficult or awkward to handle. It is desirable to strike a balance between eliminating redundancies and sacrificing clarity.

Initially the second point might be surprising, especially if one views axioms for geometry as reflecting the properties of physical space. However, since the formal setting for deductive geometry is an idealization of these properties, it is legitimate to ask for evidence that such axioms do not lead to problems like conclusions that logically contradict each other. In mathematics, the appropriate evidence is known as a ***relative consistency proof***. In other words, the objective is to prove that the setting for geometry is consistent if our standard assumptions for the arithmetic of positive whole numbers are consistent, so that a lack of consistency for the geometric axioms would imply deeper problems with our understanding of simple arithmetic. Landmark results of K. Gödel (roughly pronounced GAY – del, 1906 – 1978) show that one can never be absolutely sure that our logical setting for arithmetic is logically consistent, but for thousands of years this has proven to be a totally reliable working assumption, and in the 80 years following Gödel's work nothing has changed this despite the enormous amount of new mathematics developed during that time. The following whimsical comment due to André Weil (pronounced VAY, 1906 – 1998) summarizes this state of affairs:

> God exists since mathematics is consistent, and the Devil exists since we cannot prove it.

During the past two decades a few highly reputable mathematicians have begun to look seriously at the possibility that the generally accepted logical foundations for mathematics are not entirely consistent, but it is too early to predict what these studies will yield.

Important note. In any discussion of simplified axiom systems, it is important to remember that **one rarely gets something for nothing. Showing that some axioms are logical consequences of others will always require proofs, and many of these proofs might be extremely long, difficult, or not especially well motivated at first glance.**

As noted in the main notes for this unit, the axiom systems of both Hilbert and Birkhoff can found at the following online sites:

http://en.wikipedia.org/wiki/Hilbert%27s_axioms

http://en.wikipedia.org/wiki/Birkhoff%27s_axioms

Both of these axiom sets give priority to clarity over conciseness. In fact, the following earlier paper by Birkhoff gave an extremely economical list of axioms for Euclidean geometry:

> G. D. Birkhoff, A set of postulates for plane geometry (based on scale and protractors), *Annals of Mathematics* (2) **33** (1932), pp. 329 – 345.

Similarly, an extremely short list of axioms for Euclidean geometry based upon Hilbert's approach is given in the following book:

> H. G. Forder, ***The foundations of Euclidean geometry*** **(**Reprint of the original 1927 edition**).** Dover, New York, 1958.

The axioms described below are not quite as terse or strong as those in Birkhoff's paper, but they are in the same spirit as Birkhoff's axioms described in the online link, and they rely very substantially on certain results from Forder's book. As in Birkhoff's approach, the underlying idea is that one should use the algebraic and other properties of the real numbers to expedite the development of geometry. In keeping with the modern formulation of mathematics in terms of set theory, the axioms for plane geometry begin with a set **EPLANE** containing at least three elements. The ***points*** in this geometry will simply be the elements of the given set **EPLANE**. Euclidean plane geometry can then be based upon the following six axioms.

AXIOM 1:

There exist nonempty proper subsets of **EPLANE** called ***lines***, with the properties that each two points belong to ***exactly*** one line.

AXIOM 2:

For every pair of points **A, B** ∈ **EPLANE** there exists a ***unique*** nonnegative real number **|AB| = |BA|**, the ***distance*** from **A** to **B**, which is zero if and only if **A = B**.

AXIOM 3 (Birkhoff's Ruler Postulate):

If **L** is a line and $\mathbb{R}$ denotes the set of real numbers, there exists a ***one – to – one correspondence*** ($\mathbf{X} \leftrightarrow x$) between the points **X** ∈ **L** and the real numbers $x \in \mathbb{R}$ such that the distance between **A** and **B** satisfies

$$|\mathbf{AB}| = |a - b|$$

where $A \leftrightarrow a$ and $B \leftrightarrow b$.

Digression. Before proceeding we need to define betweenness and convexity for a system satisfying the first three axioms. Given three collinear points **A, B, C**, we shall say that **A∗B∗C** holds (equivalently, **B** is ***between*** **A** and ***C***) provided

$$|\mathbf{AC}| = |\mathbf{AB}| + |\mathbf{BC}|.$$

A subset **K** of **EPLANE** is said to be **convex** provided **A** ∈ **K**, **C** ∈ **K** and **A∗B∗C** imply **B** ∈ **K.** Equivalently, this means that if **A** and **C** lie in **K** then so does the whole closed segment joining them. Intuitively, this means that the set has no dents or holes. Illustrations in the following online sites depict some sets which are convex and others which are not:

http://www.ams.org/featurecolumn/images/cubes33.gif

http://en.wikipedia.org/wiki/Convex_set

http://www.artofproblemsolving.com/Wiki/images/a/a3/Convex_polygon.png

http://upload.wikimedia.org/wikipedia/commons/2/20/Rouleaux_triangle_Animation.gif

AXIOM 4 (Plane Separation Postulate):

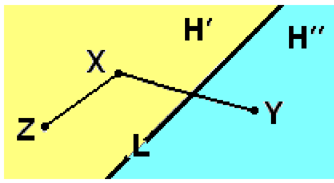
For each line **L** there are exactly two nonempty ***convex*** sets $\mathbf{H'}$ and $\mathbf{H''}$ satisfying

(*i*) **EPLANE** $= \mathbf{H'} \cup \mathbf{L} \cup \mathbf{H''}$,

(*ii*) $\mathbf{H'} \cap \mathbf{H''} = \varnothing$, $\mathbf{H'} \cap \mathbf{L} = \varnothing$, and $\mathbf{H''} \cap \mathbf{L} = \varnothing$. That is, the three sets are ***pairwise disjoint***.

(*iii*) If $\mathbf{X} \in \mathbf{H'}$ and $\mathbf{Y} \in \mathbf{H''}$ and **[XY]** is the closed segment joining **X** to **Y**, then we have $[\mathbf{XY}] \cap \mathbf{L} \neq \varnothing$.

A figure illustrating this axiom is given below. As in the statement of the third condition, one has $\mathbf{X} \in \mathbf{H'}$ and $\mathbf{Y} \in \mathbf{H''}$, while the point ***Z*** lies on the same side $\mathbf{H'}$ as **X**. Note in particular that the entire segment **[XZ]** lies in $\mathbf{H'}$.



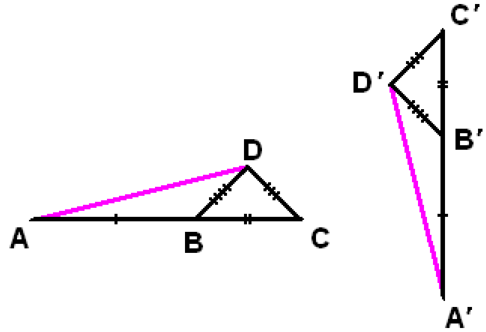
AXIOM 5 (Veblen's Planar Congruence Postulate):

Suppose that we are given two quadruples of points **A, B, C, D** and **A′, B′, C′, D′** such that the following hold:

1. **B** is between **A** and **C**.
2. **B′** is between **A′** and **C′**.
3. **D** does not lie on the line containing **A, B** and **C**.
4. **D′** does not lie on the line containing **A′, B′** and **C′**.
5. We have $|\mathbf{AB}| = |\mathbf{A'B'}|$, $|\mathbf{BC}| = |\mathbf{B'C'}|$, and $|\mathbf{CD}| = |\mathbf{C'D'}|$.

Then $|\mathbf{AD}| = |\mathbf{A'D'}|$ also holds.

The figure below illustrates Veblen's Planar Congruence Postulate; following standard practice in elementary geometry, we have marked the corresponding segments assumed to have equal length.



This formulation of a congruence postulate is due to Oswald Veblen (1880 – 1960), who made numerous contributions to the axiomatic foundations of geometry; one of his uncles was the noted economist and social critic Thorstein Veblen (1857 – 1929).

AXIOM 6 (Euclidean Parallel Postulate, Playfair's version):

> For a given line **L** and a point **P** not on **L**, there exists ***one and only one*** line **L′** through **P** such that **L′** is ***parallel*** to **L** (in other words, the lines are coplanar but do not have any points in common).

Proving that this axiom system actually implies the Birkhoff axioms in the online link is nontrivial and beyond the scope of this course; the main point here is that one can write down a set of axioms for Euclidean geometry that does not look much more complicated than the stated assumptions in the ***Elements***.

General considerations about axioms for geometry

We have already mentioned the question of relative logical consistency. There are also two other basic issues for any set of axioms. One, which is implicit in the discussion thus far, is ***logical independence*** of axioms. If we make a list of three assumptions but the third is a logical consequence of the other two, then there is no formal reason to include the third one (although there may well be other reasons for doing so). Another issue involves the ***completeness*** of a list of axioms. This means that any two systems are essentially the same for all mathematical purposes. In the case of our axioms for geometry this would mean the existence of a one-to-one correspondence between the sets of points $\mathbb{P}$ and $\mathbb{Q}$ such that collinear sets of points in $\mathbb{P}$ correspond to collinear sets of points in $\mathbb{Q}$, noncollinear sets of points in $\mathbb{P}$ correspond to noncollinear sets of points in $\mathbb{Q}$, and if the points **X** and **Y** in $\mathbb{P}$ correspond to the points **A** and **B** in $\mathbb{Q}$, then the distances between points satisfy $|AB| = |XY|$. Frequently one also says that a system of axioms is ***categorical*** if it has this sort of uniqueness property.

Certainly mathematicians want all axiomatic systems to be (relatively) logically consistent, and we have already discussed the desirability of logical independence. For Euclidean geometry one wants a categorical axiomatic system, but there are also many places in mathematics where one does not. For example, the standard rules of arithmetic for addition, subtraction and multiplication are axiomatized into an abstract concept called a ***commutative ring***, and **one important feature of these axioms is that they apply to a vast array of algebraic systems that are quite different from each other in many important respects.** We shall discuss consistency, independence and completeness for the axioms of geometry below. The following online references contain further information of a general nature:

http://en.wikipedia.org/wiki/Axiomatic_system

http://www.andrew.cmu.edu/course/80-110/math_outline.html

Proofs that the axioms for Euclidean geometry are consistent and categorical appear in the book by Moïse cited earlier in these notes (see the document on constructions by straightedge and compass). Both proofs are based upon standard material from lower and upper level undergraduate mathematics course, but the entire arguments are long and sometime messy. The proof of categoricity is the easier one, and it essentially comes down to the construction of a rectangular coordinate system (this is worked out explicitly in Moïse). Here is an online summary of the main steps in the relative consistency proof:

http://math.ucr.edu/~res/math133/verifications.pdf

The issue of independence is particularly significant because of the interest in proving Euclid's Fifth Postulate (equivalently, Axiom **6** in our list) from the others. In the early nineteenth century several mathematicians — most notably C. F. Gauss (1777 – 1855), J. Bolyai (1802 – 1860) and N. Lobachevsky (1792 – 1856) — concluded independently that one could not prove the Fifth Postulate from the others and that it was possible to work out an entire theory of geometry in which this postulate is false. Somewhat later in the same century, E. Beltrami (1835 – 1900) confirmed this rigorously by constructing an explicit system consisting of a set **HPLANE**, with a suitable notion of distance between two points and a corresponding family of abstract subsets as lines, such that our first five axioms for geometry hold but the sixth, which is equivalent to Euclid's Fifth Postulate, is false. In particular, Beltrami's research showed that the axiomatic system given by assuming

1. all of Axioms **1 – 5** as above are **true**,
2. the final Axiom **6** as above is **false**,

is (relatively) ***logically consistent***; strictly speaking, this last conclusion is basically due to F. Klein (1849 – 1925), but the explicit verification of axioms is due to Beltrami. In other words, **such an axiom system is consistent if the standard axioms for the positive integers are logically consistent; it also follows that such an axiom system is logically consistent provided the standard axioms for Euclidean geometry are logically consistent.**

Subsequently other models proving the independence of the Parallel Postulate were constructed by Klein and H. Poincaré (1854 – 1912). The book by Moïse also outlines a proof that Poincaré's model satisfies the same two conditions as Beltrami's. Additional information about this topic (***Non – Euclidean Geometry***) is given in Unit V of the notes.

Given that Axioms **1 – 6** are categorical, it is natural to ask what happens if Axioms **1 – 5** remain true but Axiom **6** is false. The answer is that the axioms are very nearly categorical. More precisely, if we are given two such systems with underlying sets of points $\mathbb{P}$ and $\mathbb{Q}$, then there is a one – to – one correspondence between the sets of points $\mathbb{P}$ and $\mathbb{Q}$ such that collinear sets of points in $\mathbb{P}$ correspond to collinear sets of points in $\mathbb{Q}$, noncollinear sets of points in $\mathbb{P}$ correspond to noncollinear sets of points in $\mathbb{Q}$, and if the points **X** and **Y** in $\mathbb{P}$ correspond to the points **A** and **B** in $\mathbb{Q}$, then there is a ***unique*** positive constant r such that the distances between points satisfy $|\mathbf{AB}| = r\,|\mathbf{XY}|$ for all choices of **A, B, X** and **Y.** A proof of this result is described in Section 7.7 of the following book:

A. Ramsay and R. D. Richtmeyer, ***Introduction to Hyperbolic Geometry.*** Springer – Verlag, New York NY, 1995.

One point to note in the uniqueness result is the need to introduce a positive constant r, which can be viewed as a "curvature constant" reflecting a choice of measurement units for lengths and areas. The uniqueness result for systems satisfying all six Euclidean axioms did not require a positive constant. One explanation of this difference is that the usual theory of similar triangles in Euclidean geometry breaks down completely if the Parallel Postulate is false. There is one choice of the curvature constant for which the key measurement formulas are particularly simple to state. In this case, the main ideas of the uniqueness proof are implicit in the following sequence of files:

http://www.ms.uky.edu/~droyster/courses/spring02/classnotes/Chapter04.pdf
http://www.ms.uky.edu/~droyster/courses/spring02/classnotes/Chapter05.pdf
http://www.ms.uky.edu/~droyster/courses/spring02/classnotes/Chapter06.pdf

Postscript — Axioms for Euclidean and non – Euclidean solid geometry

Several complications arise if one tries to construct a similar set of axioms for solid geometry. As in the planar case, we begin with a set **ESPACE**, this time containing at least **four** elements. The ***points*** in this geometry will simply be the elements of the given set **ESPACE**. One immediate complication is that, in addition to a family of subsets called ***lines***, there is also a second family of subsets called ***planes***. These must be assumed to satisfy a list of further properties in addition to **AXIOM 1;** the additional properties, which are all fairly elementary, are listed on page 36 of the following online document:

http://math.ucr.edu/~res/math133/geometrynotes02a.f13.pdf

In the 3 – dimensional setting one again has a notion of distance (**AXIOM 2**) and the Birkhoff Ruler Postulate (**AXIOM 3**) exactly as in the planar case, but the Plane Separation Postulate (**AXIOM 4**) must be augmented with a stronger assumption called the ***Space Separation Postulate***:

AXIOM 4+ (Space Separation Postulate):

For each plane **P** there are exactly two nonempty ***convex*** sets $\mathbf{H'}$ and $\mathbf{H''}$ satisfying

(i) **ESPACE** $= \mathbf{H'} \cup \mathbf{P} \cup \mathbf{H''}$,

(ii) $\mathbf{H'} \cap \mathbf{H''} = \varnothing$, $\mathbf{H'} \cap \mathbf{P} = \varnothing$, and $\mathbf{H''} \cap \mathbf{P} = \varnothing$. That is, the three sets are ***pairwise disjoint***.

(iii) If $\mathbf{X} \in \mathbf{H'}$ and $\mathbf{Y} \in \mathbf{H''}$ and **[XY]** is the closed segment joining **X** to **Y**, then we have $[\mathbf{XY}] \cap \mathbf{P} \neq \varnothing$.

It turns out that if **AXIOM 4+** is true, then one can prove that **AXIOM 4** is true for every plane in **ESPACE**. There is no change to **AXIOM 5** (however, the four point configurations **A, B, C, D** and $\mathbf{A'}, \mathbf{B'}, \mathbf{C'}, \mathbf{D'}$ do not necessarily have to lie in the same plane), and in **AXIOM 6** (the Parallel Postulate) the condition that $\mathbf{L'}$ and **L** be coplanar is essential. This completes the list of axioms for **ESPACE**. The corresponding axioms for **HSPACE** can be obtained by assuming that all the preceding axioms except for (the modified) **AXIOM 6** are true and that (the modified) **AXIOM 6** is false. Not surprisingly, proofs that these axioms are consistent and categorical are more difficult than their 2 – dimensional counterparts.

Books on the axioms of geometry

There are too many individual titles to mention here and many of them are written at a very high level, but the following recent textbook discusses different approaches to formulating axioms in a wide – ranging way at the level of undergraduate mathematics:

G. A. Venema, ***Foundations of Geometry*** (2nd Ed.). Pearson, Boston, MA, 2012.