Cutting and pasting fallacies
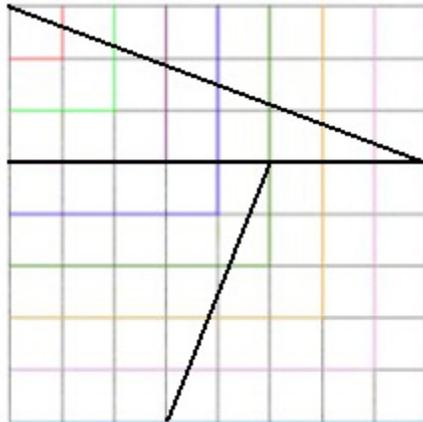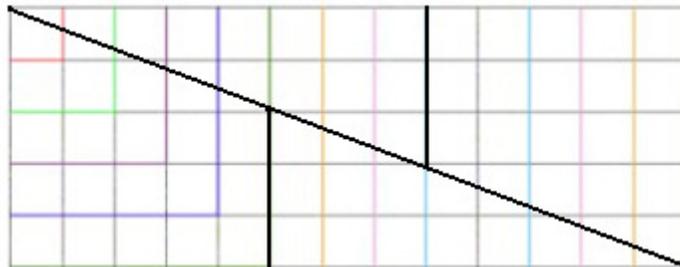
This is another example to illustrate how drawings can be misleading. The basic idea is to start with an 8 by 8 grid and decompose it into four regions as suggested below:
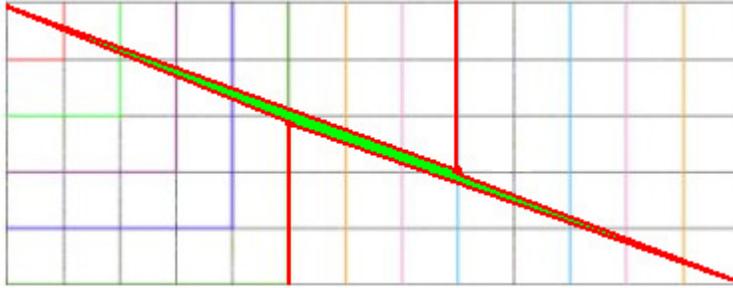

Specifically, the pieces are two right triangles such that the lengths of its perpendicular sides are 3 and 8, and a trapezoid with two right angles such that the parallel sides have lengths 5 and 8. Suppose now that we rearrange these pieces as illustrated below:

The large rectangle is 5 by 13, and therefore its area is 65 square units. However, we began with pieces coming from a decomposition of an 8 by 8 square, so the total area of the pieces, and hence the whole rectangle, should be 64 square units.

Where is the mistake?

One natural starting point would be to construct carefully drawn figures. If we do this, then we see that the four pieces do not quite fit together as claimed, but instead there is a small hole in the middle corresponding to the green parallelogram region in the figure below:

To be sure this is accurate, we should also give algebraic or analytic reasons why the second drawing above is incorrect and the third one is correct. One way of doing so is as follows: Look at the slopes of the various lines. The slope of the diagonal for the 5 by 13 rectangle is $-5/13$ (which is approximately $0.3846\ldots$), while the slope for the top of the trapezoid at the lower left is $-2/5$, and the slope for the top of the triangle at the lower right is $-3/8$. Thus we see that the diagonal for the rectangle does not contain the vertices of the triangles or the trapezoids, as suggested by the second drawing.

Another example of the same type

The following picture seems to suggest that one can rearrange the pieces from the original 8 by 8 square so that they define a figure whose area is 63 square units. The reader is invited to study this example and fine the mistake(s).