BETWEENNESS AND VECTOR ALGEBRA

If $\mathbf{a}$ and $\mathbf{b}$ are distinct points of $\mathbb{R}^n$ (where $n = 2$ or 3), then the points of the line $\mathbf{ab}$ have the form

$$\mathbf{x} \quad = \quad \mathbf{a} \ + \ t(\mathbf{b} - \mathbf{a})$$

for some uniquely determined scalar t. Given three points $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$ on $\mathbf{ab}$ with associated scalars t_1, t_2, and t_3 respectively, we shall describe the betweenness condition $\mathbf{x}_1 * \mathbf{x}_2 * \mathbf{x}_3$ in terms of t_1, t_2, and t_3.

THEOREM. *Let $\mathbf{a} \neq \mathbf{b}$ in $\mathbb{R}^n$, for $i = 1, 2, 3$ let $\mathbf{x}_i = \mathbf{a} + t_i(\mathbf{b} - \mathbf{a})$ for suitably chosen scalars t_i, and assume that the points $\mathbf{x}_i$ are distinct. Then the betweenness relation $\mathbf{x}_1 * \mathbf{x}_2 * \mathbf{x}_3$ holds if and only if* either $t_1 < t_2 < t_3$ or $t_1 > t_2 > t_3$.

Proof. We shall begin with some general remarks. If $\mathbf{x}_i = \mathbf{a} + t_i(\mathbf{b} - \mathbf{a})$ as above, then we have

$$d(\mathbf{x}_i, \mathbf{x}_j) \quad = \quad \big\| \, t_i(\mathbf{b} - \mathbf{a}) \ - \ t_j(\mathbf{b} - \mathbf{a}) \big\| \quad = \quad \big\| (t_i - t_j)(\mathbf{b} - \mathbf{a}) \big\| \quad = \quad |t_i - t_j| \cdot \|\mathbf{b} - \mathbf{a}\| \ .$$

Since $\mathbf{x}_1 * \mathbf{x}_2 * \mathbf{x}_3$ holds if and only if

$$d(\mathbf{x}_1, \mathbf{x}_3) \quad = \quad d(\mathbf{x}_1, \mathbf{x}_2) \ + \ d(\mathbf{x}_2, \mathbf{x}_3)$$

and the latter holds if and only if

$$|t_1 - t_3| \cdot \|\mathbf{b} - \mathbf{a}\| \quad = \quad |t_1 - t_2| \cdot \|\mathbf{b} - \mathbf{a}\| \ + \ |t_2 - t_3| \cdot \|\mathbf{b} - \mathbf{a}\| \ .$$

Since $\mathbf{a} \neq \mathbf{b}$, the quantity $\|\mathbf{b} - \mathbf{a}\|$ is positive, and therefore the last equation is equivalent to

$$|t_1 - t_3| \quad = \quad |t_1 - t_2| \ + \ |t_2 - t_3| \ .$$

Therefore we need to show that the preceding equation holds if and only if $t_1 < t_2 < t_3$ or $t_1 > t_2 > t_3$.

By our hypotheses the numbers t_1, t_2 and t_3 are distinct, and accordingly the conclusion can be split into two parts:

(A) *If $t_1 < t_2$, then $\mathbf{x}_1 * \mathbf{x}_2 * \mathbf{x}_3$ holds if and only if $t_2 < t_3$.*

(B) *If $t_1 > t_2$, then $\mathbf{x}_1 * \mathbf{x}_2 * \mathbf{x}_3$ holds if and only if $t_2 > t_3$.*

Case (A): We are given that $t_1 < t_2$. If $t_2 < t_3$ then

$$|t_1 - t_3| \quad = \quad t_3 - t_1 \quad = \quad (t_3 - t_2) \ + \ (t_2 - t_1) \quad = \quad |t_3 - t_2| \ + \ |t_2 - t_1|$$

and therefore we have $\mathbf{x}_1 * \mathbf{x}_2 * \mathbf{x}_3$.

To prove the converse, we shall show that if $t_2 < t_3$ is false — which in our setting means that $t_2 > t_3$ — then $\mathbf{x}_1 * \mathbf{x}_2 * \mathbf{x}_3$ does not hold. There are two subcases, depending upon whether $t_1 < t_3$ or $t_1 > t_3$. In the first subcase, if $\mathbf{x}_1 * \mathbf{x}_2 * \mathbf{x}_3$ holds, then the condition on the t_i can be rewritten as

$$t_3 - t_1 = 2t_2 - t_3 - t_1$$

which implies that $2t_3 = 2t_2$ and hence $t_3 = t_2$, which contradicts the fact that the t_i are distinct. Therefore $\mathbf{x}_1 * \mathbf{x}_2 * \mathbf{x}_3$ does not hold if $t_1 < t_2$, $t_2 > t_3$, and $t_1 < t_3$. On the other hand, if $t_1 > t_3$ then then similar considerations imply that if $\mathbf{x}_1 * \mathbf{x}_2 * \mathbf{x}_3$ holds, then the condition on the t_i can be rewritten as

$$t_1 - t_3 = 2t_2 - t_3 - t_1$$

which implies that $2t_1 = 2t_2$ and hence $t_1 = t_2$, which again contradicts the fact that the t_i are distinct. Therefore $\mathbf{x}_1 * \mathbf{x}_2 * \mathbf{x}_3$ also does not hold if $t_1 < t_2$, $t_2 > t_3$, and $t_1 > t_3$. Combining these, we see that $\mathbf{x}_1 * \mathbf{x}_2 * \mathbf{x}_3$ does not hold if $t_1 < t_2$, and $t_2 > t_3$. Therefore if $\mathbf{x}_1 * \mathbf{x}_2 * \mathbf{x}_3$ holds and $t_1 < t_2$, then we must also have $t_2 < t_3$.

Case (B): We are given that $t_1 > t_2$. If we let $u_i = -t_i$, then the conditions

$$|t_1 - t_3| = |t_1 - t_2| + |t_2 - t_3| , \quad |u_1 - u_3| = |u_1 - u_2| + |u_2 - t_3|$$

are equivalent to each other, and we have $u_1 < u_2$. Since the numbers u_i are distinct if and only if the numbers t_i are, we can now use the argument in Case (A) to show that the displayed equations hold if and only if $u_2 < u_3$. If we translate this back into a statement about the t_i, we conclude that if $t_1 > t_2$ then the displayed equation holds if and only if $t_2 > t_3$.■

This result allows us to translate statements about betweenness of collinear points into inequality statements involving real numbers and vice versa.