Conics, congruence and similarity

Several documents in this directory, most notably

http://math.ucr.edu/~res/math133/extreme-pts.pdf

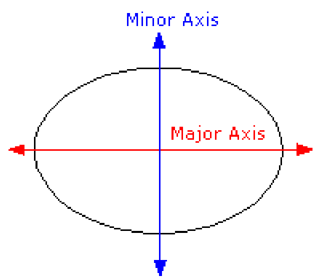
state and prove results confirming that many polygonal subsets of the coordinate plane are not congruent in the sense of the course notes; in most if not all cases it seems clear from looking at the figures that they cannot be congruent, and the results in the cited document (and others of the same type) verify that our intuition is correct. Clearly one can consider similar questions for conic section curves in the coordinate plane, which can be defined by certain nontrivial quadratic equations in two variables. In particular, if we take the four standard classes of conic sections

circles*, *ellipses*, *parabolas*, *hyperbolas

then we expect that no curve in one class is congruent or similar to a curve in a different class and that within all classes except the first the curves can have many different sizes and shapes (all circles have the same shape, but their sizes may differ). Formally, we expect that two conics are similar if they have the same shape and that they are congruent if they also have the same size. The simplest cases to analyze are circles and ellipses, so we shall concentrate upon them here and state the main results:

> *Given two circles, each is similar to the other. Two circles are congruent if and only if their radii are equal in length.*
> *Given two ellipses, they are congruent if and only if the lengths of their major axes and the lengths of their minor axes are equal. Two ellipses are similar if and only if the ratios of the lengths of the minor axes to the lengths of the major axes are equal.*

For the sake of convenience here is a drawing of an ellipse along with its major and minor axes.



(***Source:*** http://jkepler33.blogspot.com/2011/04/funny-story-about-ellipses_11.html)

The proofs of the two displayed results (and their counterparts for parabolas and hyperbolas) require a considerable amount of work involving geometrical transformations of the sorts described in Sections **II.4** and **III.5** of the class notes, and the level of the argument is slightly higher than that of the course. Statements and proofs of the classifications for all four types of curves appear in the last part of the document

http://math.ucr.edu/~res/progeom/quadrics3.pdf (especially pages $10-16$) in the series listed below. We should note that the proofs use material on the earlier pages of the cited document as well as background from **http://math.ucr.edu/~res/progeom/quadrics1.pdf** in the same series. The lecture notes

http://math.ucr.edu/~res/progeom/pg-all.pdf

describes the basic setting for the methods and results in the documents on quadrics.

http://math.ucr.edu/~res/progeom/quadrics0.pdf

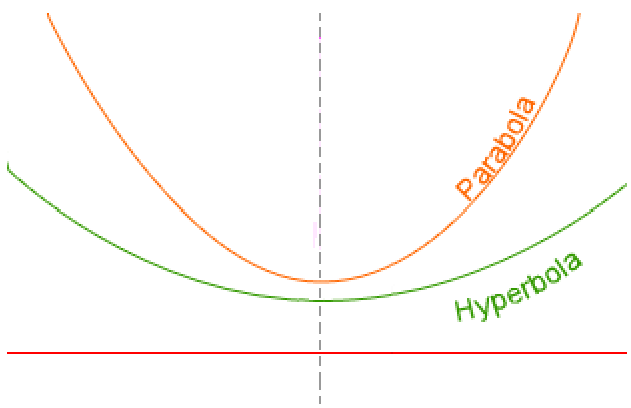
http://math.ucr.edu/~res/progeom/quadrics1.pdf

http://math.ucr.edu/~res/progeom/quadrics2.pdf

http://math.ucr.edu/~res/progeom/quadrics3.pdf

http://math.ucr.edu/~res/progeom/quadrics4.pdf

Finally, one easy way of seeing that a parabola and hyperbola are not congruent or similar is to note that a parabola has only one branch but a hyperbola has two (in formal mathematical terms, a parabola is a connected subset of the plane but a hyperbola is not). A more substantial question is whether a parabola can be congruent or similar to ***one branch*** of a hyperbola. If one looks at these curves carefully, it appears that a parabola and a branch of a hyperbola will never be congruent to each other. For example, one might expect this because a hyperbola has asymptotes but a parabola does not.



http://www.formyschoolstuff.com/school/math/glossary/images/eccentricity.gif

The document **http://math.ucr.edu/~res/progeom/quadrics4.pdf** gives two proofs that a parabola and a branch of a hyperbola are neither congruent nor similar (in fact, such curves are not even equivalent in a weaker sense known as affine equivalence). The first proof is relatively short but requires nontrivial input from projective geometry and point set topology, and the second, which is elementary but longer, is based upon the differences between the asymptotic properties of tangent lines to such curves which were mentioned in the preceding paragraph.