TANGENTS TO AN ELLPISE THROUGH AN EXTERNAL POINT

We have noted that the concept of geometrical transformation was implicit in the classical Greek idea of proof by superposition. Our purpose here is to show how such transformations can yield relatively simple proofs of some geometrical results.

BACKGROUND MATERIAL. As the title suggests, we are interested in questions involving tangent vectors to ellipses in $\mathbf{R}^2$. As such, we shall use the standard description of tangents in terms of differential calculus, and we shall also use the representability of ellipses and other curves by parametric equations. The starting point is the following observation, which can be proved by direct calculation:

PROPOSITION. *Let $\mathbf{y}(t)$ be a parametrized curve defined on an open interval (c,d) with values in $\mathbf{R}^2$, and assume that $\mathbf{y}(t)$ has a continuous derivative $\mathbf{y}'(t)$. Suppose further that we are given an affine transformation of $\mathbf{R}^2$ defined by $F(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$, where A is an invertible 2×2 matrix and $\mathbf{b} \in \mathbf{R}^2$, and let $\mathbf{z}$ be the curve $\mathbf{z}(t) = F\big(\,\mathbf{y}(t)\,\big)$. Then we have $\mathbf{z}'(t) = A\,\mathbf{y}'(t)$.*

Verification is left as an exercise to the reader; one can use the standard coordinate forms of vectors and matrices in which vectors are viewed as 2×1 matrices, the vector $\mathbf{w}$ has coordinates (w_1, w_2), and the linear transformation associated to A sends this vector to the one with coordinates

$$\big(\,a_{1,1}w_1 + a_{1,2}w_2,\; a_{2,1}w_1 + a_{2,2}w_2\,\big)\ .$$

Here is a statement of the main result.

THEOREM. *Let $\Gamma \subset \mathbf{R}^2$ be an ellipse, and let $\mathbf{p} \in \mathbf{R}^2$ be an external point. Then there are two tangents to Γ which pass through $\mathbf{p}$.*

Our proof will use the results in Section V.2 (pp. 80–83) of the following online linear algebra notes:

`http://math.ucr.edu/~res/math132/linalgnotes.pdf`

In particular, these results imply that if Γ is an arbitrary ellipse in the coordinate plane, then there is an affine transformation (in fact, a rigid motion or Galilean transformation) F such that F maps Γ to a standard ellipse Γ_1 of the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} \;=\; 1$$

where $a, b > 0$. We have not been precise about the notion of external point for Γ, but one way to describing the external points of this ellipse is to say that they are the points which map under F to the obvious external set of Γ_1 consisting of all points for which

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} \;>\; 1\ .$$

By the proposition, it is enough to prove the theorem for ellipses defined by the standard equations as above.

One can take this another step further and reduce the proof to the case where Γ_1 is the standard unit circle Γ_0 defined by $x^2 + y^2 = 1$, for if G is the affine transformation such that

$$G(w_1, w_2) \quad = \quad G(w_1/a_1,\, w_2/a_2)$$

then G maps the previously described Γ_1 to Γ_0 and sends external points for the curve Γ_1 into external points for Γ_0.

Thus we have reduced everything to a special case of a basic result in classical Euclidean geometry: *If we are given a circle C and an external point $\mathbf{p}$, then there are exactly two tangents from $\mathbf{p}$ to C.*

For the sake of completeness, here is a proof using vectors. It requires to pieces of background information:

(1) *Suppose we are given ΔABC in the plane such that the midpoint D of $[AC]$ is equidistant from the three vertices. Then $\angle ABC$ is a right angle.*

(2) *If C is a circle and L is a line through the point $\mathbf{p} \in C$, then L is tangent to C at $\mathbf{p}$ if and only if L is perpendicular to the line joining $\mathbf{p}$ to the center of C.*

Both of these can be proven using vectors. In the first case, one starts with the equations $D = \frac{1}{2}(A + C)$ and $|D - A|^2 = |D - B|^2 = |D - C|^2$ and uses them to show that the dot product $\langle A - B,\, C - B\rangle$ is equal to zero. In the second case, one has a parametrization $\mathbf{x}(t)$ such that $|\mathbf{x}|^2 = r^2$ for some fixed $r > 0$, and simple differentiation shows that $2\,\mathbf{x}(t) \cdot \mathbf{x}'(t) = 0$, so that direction vectors for the two lines — which are $\mathbf{x}'(t)$ for the tangent and $\mathbf{x}(t)$ for the line through the center — must be perpendicular.

It follows (as in a classical Greek geometrical construction) that if the circle $|x|^2 = r^2$ meets the circle $|\mathbf{x}-\mathbf{p}|^2 = |\mathbf{p}|^2$ in two points $\mathbf{q}_1$ and $\mathbf{q}_2$, then the lines $\mathbf{p}\mathbf{x}_i$ are perpendicular to the lines $\mathbf{0}\mathbf{x}_i$ by (1) above, and hence by (2) the lines $\mathbf{p}\mathbf{x}_i$ are tangent to the circle at the points $\mathbf{x}_i$. To complete the proof, we need to show that the system of equations

$$|\mathbf{w} - \mathbf{p}|^2 \quad = \quad |\mathbf{p}|^2\;, \qquad |\mathbf{w}|^2 \quad = \quad r^2$$

has two solutions. Our assumption on $\mathbf{p}$ means that $|\mathbf{p}|^2 > r^2$, and hence if we write $\mathbf{p} = (a, b)$ then after some algebraic manipulation we obtain a system of two scalar equations:

$$w_1^2 \;+\; w_2^2 \quad = \quad r^2, \qquad 2(a\,w_1 \;+\; b\,w_2) \quad = \quad r^2$$

We can solve the second equation for one of w_1 or w_2 in terms of the other coordinate because a and b are not both zero, and if we substitute the resulting expression into the first equation we obtain a quadratic equation in one of the coordinates. The condition $|\mathbf{p}|^2 = a^2 + b^2 > r^2$ implies that this equation has two real roots, and it follows that there are exactly two points which lie on the two circles.■