EXTREME POINTS AND AFFINE EQUIVALENCE

The purpose of this note is to use the notions of extreme points and affine transformations — which are studied in the file `affine-convex.pdf` — to prove that certain standard geometrical figures are not affine equivalent (and *a fortiori* not congruent). In particular, these results imply that a triangle and a convex quadrilateral cannot be affine equivalent and hence cannot be congruent. On the intuitive level, the conclusions are fairly self-evident; however, the proofs yield more general insights into the geometric properties of figures and into criteria for determining whether more general pairs of subsets are congruent or affine equivalent.

Needless to say, we shall use material from `affine-convex.pdf` as needed.

Definition. Let $n \geq 3$, and let $V = \{A_1, \ \dots \ , A_n\}$ be a linearly ordered set of points in $\mathbb{R}^2$ such that no three are collinear; for the sake of notational convenience we shall also denote A_n by A_0. We shall say that the broken line curve Γ_V defined by

$$[A_1A_2] \ \cup \ [A_2A_3] \ \cup \ \dots \ \cup \ [A_nA_1]$$

is a *convex polygon* if for every k such that $0 \leq k < n$ all points in $V - \{A_k, A_{k+1}\}$ lie on the same open half-plane determined by the line A_kA_{k+1}. The elements of V are called the *vertices* of Γ_V.

Reminder. If $L \subset \mathbb{R}^2$ is a line defined by the linear equation $g = 0$, then (i) the two open half-spaces determined by L are the sets determined by the strict inequalities $g > 0$ and $g < 0$, (ii) the two closed half-spaces determined by L are the sets determined i by the non-strict inequalities $g \geq 0$ and $g \leq 0$.

If the number of vertices is unknown or unimportant, we shall often simply say that Γ_V is a *convex polygon*.

Remarks.

1. If $n = 3$ then the condition is true for all choices of V, and the resulting curve is just a triangle. If $n = 4$ the defining condition is satisfied if V is the set of vertices for the unit square $[0,1] \times [0,1]$, but it is not satisfied if V is the set whose elements are $(0,1)$, $(0,0)$, $(1,0)$ and $\left(\frac{1}{3}, \frac{1}{3}\right)$.

2. For certain small values of n (say, less than 100) we often use the standard words to describe n-gons (*e.g.*, quadrilateral, pentagon, hexagon, octagon, decagon, *etc.*).

3. The definition itself does not address the question of whether the underlying set of points can also be written in the form

$$\Gamma_W \quad = \quad [B_1B_2] \ \cup \ [B_2B_3] \ \cup \ \dots \ \cup \ [B_mB_1]$$

for a second set of linearly ordered vertices $W = \{B_1, \ \dots \ , B_m\}$. In this note we shall only need to know that the underlying unordered sets are the same. An elementary but somewhat tedious argument also shows that the ordering is unique up to permutations of $\{1, \ \dots \ , n\}$ generated by the cyclic permutation ($k \to k+1$ for $k < n$ and $n \to 1$) and the order-reversing permutation sending k to $n + 1 - k$ for all k, but we shall not prove or use this.

4. The results of `affine-convex.pdf` imply that if F is an affine transformation and Γ_V is a convex polygon, then $F\,[\Gamma_V]$ is the convex polygon $\Gamma_{F[V]}$. — Specifically, the results of `affine-convex.pdf` show that (i) if $V = \{A_1, \ \dots \ , A_n\}$ is a linearly ordered set of points in $\mathbb{R}^2$ such that no three are collinear then so is $F[V]$, (ii) if all the points in $V - \{A_k, A_{k+1}\}$ are on the same side of A_kA_{k+1} then all the points in $F\,[V - \{A_k, A_{k+1}\}]$ are on the same side of $F(A_k)F(A_{k+1})$, (iii) an affine transformation sends the closed segment joining X and Y to the closed segment joining $F(X)$ and $F(Y)$.

5. It is important to notice that *a convex polygon is* NOT *a convex set*. For example, if X is the midpoint of $[A_2A_3]$ and Y is the midpoint of $[A_1X]$, then $Y \notin \Gamma_V$. This is immediately apparent from a simple drawing when $n = 3$ (see Figure 1 at the end of this file), but for the sake of completeness we shall give a rigorous proof (which may be skipped without loss of continuity).

> PROOF. Assume that $Y \in \Gamma_V$ and in particular that $Y \in [A_kA_{k+1}]$, where $0 \le k < n$. First of all, observe that $k \ne 1$, for if this were true then $Y \in A_1A_2$ would imply that $A_1X = A_1Y = A_1A_2$, so that $\{A_1, A_2, X\}$ is collinear; since $\{A_2, A_3, X\}$ is collinear by construction, this implies that $A_3 \in A_1A_2$, which contradicts our assumption that no three vertices are collinear. This shows that $k \ne 1$. — Next, $k = 0$ would imply that $\{A_n, Y, A_1\}$ is collinear, which in turn implies that $X \in [A_nA_1]$; since X is the midpoint of $[A_2A_3]$ and no three vertices are collinear, it would follow that $A_2A_3 \ne A_nA_1$ and hence A_2 and A_3 lie on opposite sides of A_nA_1, contradicting our basic assumption about the vertices. Therefore we also have $k \ne 0$. — Similarly, $k = 2$ would imply that $\{Y, A_2, A_3\}$ is collinear, and since $X \in (A_2A_3)$ this line also contains X. We can now use $A_1 * Y * X$ to conclude that $A_1 \in A_2A_3$, again contradicting our basic assumption about the vertices, and thus we further have $k \ne 2$. — Finally, if $n \ge 4$ we need to exclude the cases where $3 \le k < n$. In this case $Y \in A_kA_{k+1}$ would imply that A_1 and X lie on opposite sides of A_kA_{k+1}, and $X \in (A_2A_3)$ implies that X and A_2 lie on the same side of A_kA_{k+1} (regardless of whether or not $k = 3$). Combining these, we find that A_1 and A_2 lie on opposite sides of A_kA_{k+1}, once again contradicting our basic assumption about the vertices, and thus showing that $Y \notin [A_kA_{k+1}]$ the remaining values of k.■

Extreme points of regular polygons

The notion of extreme point is meaningful for an arbitrary subset S of $\mathbb{R}^n$ (*i.e.*, the point is not between two other points of S). Then as in `affine-convex.pdf` we can conclude that *if F is an affine transformation of $\mathbb{R}^n$ and $S \subset \mathbb{R}^n$, then F sends extreme points of S to extreme points of $F[S]$.*

We can now state the main results.

THEOREM 1. *If V is the ordered set of vertices for the convex polygon Γ_V, then V is the set of extreme points for Γ_V.*

COROLLARY 2. *A convex m-gon and a convex n-gon cannot be affine equivalent (and hence cannot be congruent) if $m \ne n$.*

Proof that Theorem 1 implies Corollary 2. The theorem implies that if Γ_V and Γ_W are convex n- and m-gons respectively, then their extreme points are V and W respectively. Therefore if F is an affine transformation mapping Γ_V to Γ_W, then F sends V to W by the earlier remark on extreme points and affine transformations. But this means that V and W must have the same numbers of elements, sot htat $n = m$.■

The following observation plays a key role in the proof of Theorem 1:

LEMMA 3. *Let $n \ge 3$, and let $V = \{A_1, \ldots, A_n\}$ be the linearly ordered vertices of a convex n-gon Γ_V (as usual, take $A_0 = A_n$). Then for each k such that $0 \le k < n$ we have $A_kA_{k+1} \cap \Gamma_V = [A_kA_{k+1}]$.*

Proof of Lemma 3. Let g be a linear function whose zero set is the line $A_k A_{k+1}$; multiplying g by -1 if necessary, we may assume that $g(A_{k+2}) > 0$, where we let $A_{n+1} = A_1$ when $k = n-1$. It follows immediately that $g(A_i) > 0$ for all $i \neq k, k+1$ (since all these points lie on the same side of $A_k A_{k+1}$), and it will suffice to verify that g is positive on each open interval $(A_j A_{j+1})$ for $j \neq k$.

Suppose now that $X \in (A_j A_{j+1})$ for some $j \neq k$, so that $X = sA_j + tA_{j+1}$ where $s, t > 0$ and $s + t = 1$. Since $j \neq k$ at least one of $g(A_j)$ and $g(A_{j+1})$ must be positive; in any case we know that both are nonnegative. These imply that

$$g(X) \quad = \quad s \cdot g(A_j) \ + \ t \cdot (A_{j+1}) \quad > \quad 0$$

so that X does not lie on $A_k A_{k+1}$. By construction none of the vertices A_j (where $j \neq k$) lie on the latter line, and thus the only points in $A_k A_{k+1} \cap \Gamma_V$ are the points in $[A_k A_{k+1}]$.■

Proof of Theorem 1. First of all, we claim that the set of extreme points is contained in the set V of vertices, for each point in an open edge $(A_k A_{k+1})$ lies between the vertices A_k and A_{k+1}. Therefore it remains to prove that every vertex is an extreme point.

Let A_k be a typical vertex, and let g be the linear function described in Lemma 3 for the line $A_k A_{k+1}$. Observe that g is nonnegative on the vertices, and therefore on each closed edge $[A_j A_{j+1}]$ the linear function g is nonnegative.

If we have $A_k = sY + tZ$ for some $Y, Z \in \Gamma_V$ where $s, t > 0$ and $s + t = 1$, then the observation in the preceding paragraph implies that

$$0 \quad = \quad g(A_k) \quad = \quad s \cdot g(Y) \ + \ t \cdot g(Z)$$

and since $g(Y), g(Z) \neq 0$ this can only happen if $g(Y) = g(Z) = 0$; *i.e.*, both Y and Z must lie on the line $A_k A_{k+1}$, and by Lemma 3 these points actually lie on the closed segment $[A_k A_{k+1}]$. Since A_k is not between two points on this closed segment, it follows that Y and Z cannot exist, and hence A_k must be an extreme point of Γ_V.■

Solid polygonal regions

We shall now prove a similar result for the interior regions of convex polygons. Still using the same notation as before, let g_k be the linear function such that its zero set is the line $A_k A_{k+1}$ and g_k is positive on all the vertices except A_k and A_{k+1}. Then we may define the solid or closed interior region of Γ_V to be the set of points X such that $g_k(X) \geq 0$ for all k. If we let G_j denote the closed half-plane on which g_j is nonnegative (where $0 \leq j < n$), then S is just the intersection $\cap_j \, G_j$.

The results of `affine-convex.pdf` now imply that an affine transformation F sends the solid interior region for Γ_V into the solid interior region for $\Gamma_{F[V]}$ (compare Remark 4 following the definition of a convex polygon). This in turn leads to the following result:

THEOREM 4. *In the preceding notation, let S denote the solid polygonal region determined by the convex polygon Γ_V. Then V is the set of extreme points for S.*

This leads immediately to the following analog of Corollary 2.

COROLLARY 5. *Let S and S' denote the closed polygonal regions associated to an n-gon and m-gon respectively. If S and S' are affine equivalent, then $m = n$.*

The corollary follows from the theorem because an affine equivalence from S to S' must send extreme points to extreme points and nonextreme points to nonextreme points by the results in `affine-convex.pdf`.■

Proof of Theorem 4. For the sake of giving a unified approach to several cases, we shall set A_{-1} and A_{n+1} equal to A_{n-1} and A_1 respectively, with similar conventions for the sets G_j (also let $G_0 = G_n$).

The first step is to prove that for each k satsifying $0 \leq k < n$ we have $A_kA_{k+1} \cap S = [A_kA_{k+1}]$. By hypothesis we know that A_k and A_{k+1} lie in G_j for every j, and therefore by convexity the closed segment $[A_kA_{k+1}]$ is contained in $\cap_j\, G_j = S$. Conversely, since $A_k \in A_{k-1}A_k$ and $A_{k+1} \in G_{k-1}$ we know that $A_kA_{k+1} \cap G_{k-1}$ is the closed ray $[A_kA_{k+1}$, and similarly since $A_{k+1} \in A_{k+1}A_{k+2}$ and $A_k \in G_{k+1}$ we know that $A_kA_{k+1} \cap G_{k+1}$ is the closed ray $[A_{k+1}A_k$. Combining these observations, we have

$$A_kA_{k+1} \;\cap\; S \;\;\subset\;\; A_kA_{k+1} \;\cap\; G_{k-1} \;\cap\; G_{k+1} \;\;=\;\; [A_kA_{k+1} \;\cap\; [A_{k+1}A_k \;\;=\;\; [A_kA_{k+1}]$$

and if we combine this with the second sentence of the paragraph we conclude that $A_kA_{k+1} \cap S = [A_kA_{k+1}]$, which is what we wanted to prove (the data are depicted in Figure 2).

By the preceding paragraph we know that $S - \Gamma_V$ is the set of all points where $g_j > 0$ for all j. We claim that none of the points in this relative complement are extreme points of S. — By continuity, if $g_j(X) > 0$ for all j then there is some $\delta > 0$ such that $|Y - X| < \delta$ implies that $g_j(Y) > 0$ for all j. If we take $Y_\pm$ to be the point $X + \frac{1}{2}\,\delta\,\mathbf{u}$ where $\mathbf{u}$ is some unit vector, then $Y_\pm \in S$ and X is the midpoint of $[Y_-Y_+]$ and therefore X is not an extreme point of X.

Since $\Gamma_V \subset S$, it follows that if $X \in \Gamma_V$ is an extreme point for S then it is also an extreme point for Γ_V. By the preceding paragraph we know that all extreme points for S are contained in Γ_V, and accordingly we know that an extreme point of S must be a vertex of Γ_V. To complete the proof, we need to show that each vertex is indeed an extreme point of S. This can be verified by the same sort of argument employed in the proof of Theorem 1 (the most significant point is that the function g_k is nonnegative on all of S.■

In particular, Corollary 5 implies that the closed polygonal regions defined by triangles and convex quadrilaterals cannot be affine equivalent or congruent.

Inequivalence of solid polygonal and circular regions

It is also intuitively apparent that the solid closed region defined by a convex polygon is not affine equivalent to the solid closed region defined by a circle. We shall conclude by presenting one approach to proving this fact (see Theorem 8 below). Once again, the goal of obtaining more insight into the structure of convex sets is at least equally important as the goal of proving the inequivalence statement.

Definition. Let $K \subset \mathbb{R}^2$ be a convex set, let $L \subset \mathbb{R}^2$ be a line, and let $p \in K \cap L$. Then L is said to be a *supporting line* for K if $K \subset L$ or all points of K lie on **exactly one** of the closed half-planes determined by L.

The lines containing the edges of a solid polygonal region S are supporting lines for S; of course there are other supporting lines which meet the vertices in just one point. In Proposition 7 we shall prove that if D is a solid circular region, then the supporting lines are the same as the tangent lines to the boundary points (similar statements are true for ellipses and parabolas, but we shall not try to prove these facts).

The next result implies that supporting lines are well-behaved with respect to affine transformations.

PROPOSITION 6. *Let $K \subset \mathbb{R}^2$ be convex, let L be a supporting line for K at $p \in K$, and let T be an affine transformation of $\mathbb{R}^2$. Then $T[L]$ is a supporting line for $T[K]$ at $T(p)$.*

Recall that if L is a line in $\mathbb{R}^n$ and T is an affine transformation, then the image $T[L]$ is also a line.

Proof. If $K \subset L$ this follows immediately, so suppose for the rest of this proof that K contains some point $q \notin L$.

Let H be the closed half-plane determined by L for which $K \subset H$, so that $q \in H - L$. Then T maps H to the closed half-plane H' containing the point $T(q)$ by the results in `affine-convex.pdf`, and therefore $T[K]$ is contained in $T[H] = H'$.■

The preceding result will allow us to simplify the computations in the proof of the result on supporting lines for closed circular regions.

PROPOSITION 7. *Let D be a solid circular region in $\mathbb{R}^2$ (in other words, all points X such that $|X - C| \leq r$ for some $C \in \mathbb{R}^2$ and $r > 0$). Then a line L is a supporting line for D if and only if it is a tangent line to the boundary circle Γ at some point of the latter, and each supporting line meets D at exactly one point.*

For the sake of completeness, we shall formally define the boundary circle Γ of D (as above) by the equation $|X - C| = r$.

Proof. The conclusion can be split into two parts as follows: First, if $|X - C| < r$ then there are no supporting lines passing through X. Second, if X lies on the boundary circle Γ, then there is exactly one supporting line, and it is the tangent line to Γ at X. Since there is a translation which maps D to a circular region of radius r whose center is the origin, by Proposition 6 it will suffice to prove the result when $C = \mathbf{0}$. Similarly, for every nonzero point in $\mathbb{R}^2$ there is a rotation sending that point to some point $(0, s)$ on the y-axis with $s \geq 0$, by another application of Proposition 6 it will suffice to prove the result when $C = \mathbf{0}$ and the point $X \in D$ has coordinates $(0, s)$ where $s \geq 0$.

STEP 1. Suppose $|X| < r$, so that $0 \leq s < r$. Let M be a line passing through X.

Subcase 1.1*:* Suppose that M is a vertical line, so that it is defined by the equation $x = 0$. Then the points $(\pm r, 0)$ lie in D but are on opposite sides of M, and hence the vertical line through X is not a supporting line.

Subcase 1.2*:* Suppose that M is not a vertical line, so that it is defined by an equation of the form $y = mx + s$ (the constant term must be s because $y(0) = s$). Then the points $(0, \pm r)$ lie in D but are on opposite sides of M, and hence the line M through X is not a supporting line.

STEP 2. Suppose $|X|$ is the point $(0, r)$. Let M be a line passing through X.

Subcase 2.1*:* Suppose that M is a horizontal line, so that it is defined by the equation $y = 0$. Then M is the tangent line to Γ at X, and it is also a supporting line because there is a unique solution to the system of equations $x^2 + y^2 = r^2$ and $y = r$; namely, $(x, y) = (0, r)$.

Subcase 2.2*:* Suppose that M is a vertical line, so that it is defined by the equation $x = 0$. The reasoning in Subcase 1.1 shows that M is not a supporting line for D, *Subcase* 2.3*:* Suppose that M is neither a horizontal nor a vertical line, so that it is defined by an equation of the form $y = mx + r$ where $m \neq 0$ (the constant term must be s because $y(0) = r$). Then the proof that M is not a supporting line reduces to verifying the following statement:

CLAIM. *There is some $h > 0$ such that each open half-plane determined by M contains points of the form $\left(x, \sqrt{r^2 - x^2}\right)$ with $|x| < h$.*

In particular, Γ contains points on each open half-plane and hence M cannot be a supporting line for D. There is a drawing of a typical example in Figure 3.

VERIFICATION OF THE CLAIM. Since the open half-planes determined by M are defined by the inequalities $y < mx + r$ and $y > mx + r$, another way of stating the claim is that there are real numbers u, v such that $0 < |u|, |v| < h$ and

$$\sqrt{r^2 - u^2} \ < \ mu \ + \ r \ , \qquad \sqrt{r^2 - v^2} \ > \ mv \ + \ r \ .$$

Since $r > 0$, by continuity we can take h so small that $mx + r > 0$ for $|x| < h$. If we do so, then the displayed inequalities are equivalent to their squares

$$r^2 - u^2 \ < \ (mu + r)^2 \ = \ m^2u^2 + 2mru + r^2 \ , \qquad r^2 - v^2 \ > \ (mv + r)^2 \ = \ m^2v^2 + 2mrv + r^2$$

which we can rewrite in the form

$$(m^2 + 1)v^2 \ + \ 2mrv \ < \ 0 \ < \ (m^2 + 1)u^2 \ + \ 2mru \ .$$

Let $\varphi(x) = (m^2 + 1)x^2 + 2rmx$, so that the preceding display can be rewritten as $\varphi(v) < 0 < \varphi(u)$. By construction $\varphi(0) = 0$ and $\varphi'(x) = 2rm + 2(m^2 + 1)x$, so that $\varphi'(0) \neq 0$ and by continuity there is some $h > 0$ suc that the signs of $\varphi'(x)$ and $\varphi'(0)$ are the same when $|x| < h$. This sign is positive or negative, depending upon whether m is positive or negative, so that φ is strictly increasing or decreasing for $|x| < h$.

If $m > 0$ then φ is strictly increasing, and hence if $-h < u < 0 < v < h$ we have $\varphi(u) < 0 < \varphi(v)$, which is the conclusion we wanted. On the other hand, if $m < 0$ then φ is strictly decreasing, and hence if $-h < v < 0 < u < h$ we have $\varphi(u) < 0 < \varphi(v)$, which is again the conclusion we wanted. Therefore we have shown that Γ has points on both of the open half-planes determined by M, and accordingly it cannot be a supporting line for D.■

THEOREM 8. *A solid polygonal region and a solid circular region cannot be affine equivalent.*

Proof. Suppose that T is an affine transformation mapping the solid polygonal region S to the solid circular region D. Let $X \in P$ be such that X has a supporting line L for which $L \cap S$ is a closed interval. By Proposition 6 the line $T[L]$ is a supporting line for $D = T[S]$ at $T(X)$ a such that $T[L] \cap D = T[L \cap S]$ is also a closed interval. On the other hand, Proposition 7 implies that no such supporting lines exist in D, and therefore there is no affine transformation T sending S to D.■

DRAWINGS TO ACCOMPANY THE FILE

Extreme points and affine equivalence

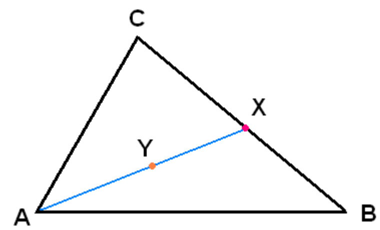

Figure 1

A convex set which contains Δ**ABC** will also contain the midpoint **Y** of the segment **[AX],** where **X** is the midpoint of edge **[BC].** The notes prove the "visually obvious" fact that **Y** does not belong to the triangle**,** and therefore the triangle is not a convex set**.**

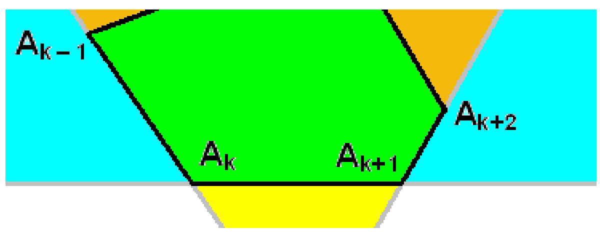

Figure 2

The solid polygonal region is colored in green with a black border**.** It is contained in the intersection of

the closed half – plane determined by the line $A_{k-1}A_k$ and the external point A_{k+1} with

the closed half – plane determined by the line $A_{k+1}A_{k+2}$ and the external point A_k**.**

One key step in the proof of Theorem 4 is to show that the intersection of the line A_kA_{k+1} with the closed polygonal region is equal to the closed segment $[A_kA_{k+1}]$**,** and the illustration suggests that the intersection with the two closed half – planes is equal to that closed segment (hence the intersection of the line with the closed polygonal region is contained in the segment)**.**

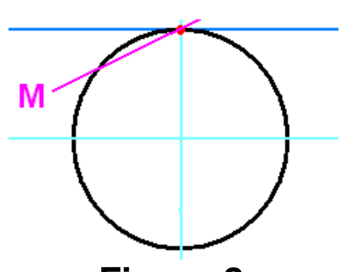

Figure 3

The line **M** passes through the top **(red)** point of the circle but is not tangent to the circle**.** Note that there are points of the circle on both half – planes determined by **M;** in fact**,** there are points of both types in a small arc centered at the top point**.**