The distance between two skew lines

We shall use vector geometry to prove the following basic result on **skew lines**; *i.e.*, lines in $\mathbb{R}^3$ which have no points in common but are not parallel (hence they cannot be coplanar).

THEOREM. *Let L and M be two skew lines in $\mathbb{R}^3$, and for $\mathbf{x} \in L$ and $\mathbf{y} \in M$ let $d(\mathbf{x}, \mathbf{y})$ denote the distance between $\mathbf{x}$ and bf y. Then the function $d(\mathbf{x}, \mathbf{y})$ takes a positive minimum value, and if $\mathbf{x}_m$ and $\mathbf{y}_m$ are points where $d(\mathbf{x}, \mathbf{y})$ is minimized, then the line joining $\mathbf{x}_m$ and $\mathbf{y}_m$ is perpendicular to both L and M.*

In classical Euclidean geometry this is usually stated in the form, "The shortest distance between two skew lines is along their common perpendicular." Not surprisingly, it is possible to prove this result using the methods of classical synthetic geometry, and nearly all the textbooks on solid geometry from the first two thirds of the 20^{th} century contain proofs of this result.

We shall use the results on the cross product $\mathbf{a} \times \mathbf{b}$ and triple product $[\mathbf{a}, \mathbf{b}, \mathbf{c}]$ (from Section I.2 of the course notes) at several points in the proof of the theorem.

Proof. There are three main parts to the argument:

(1) Proving that the distance function has an absolute minimum; under the hypotheses, we know that this minimum distance must be positive.

(2) Deriving an algebraic formula for the minimum distance.

(3) Showing that the the minimum value is realized by points $\mathbf{x}_m$ and $\mathbf{y}_m$ such that the line $\mathbf{x}_m\,\mathbf{y}_m$ is perpendicular to both L and M.

FIRST STEP. We begin by translating the problem into a question about vectors. Suppose that the skew lines have parametric equations of the form

$$\mathbf{p}_0 \;+\; t\,\mathbf{u} \quad , \quad \mathbf{p}_1 \;+\; s\,\mathbf{v}$$

where $\mathbf{u}$ and $\mathbf{v}$ are nonzero and in fact must be *linearly independent*; for if $\mathbf{u}$ and $\mathbf{v}$ are linearly dependent then the two lines described above are identical or parallel. In effect the problem is to show that the function $f(s,t) \;=\; |\,\mathbf{r}(s,t)\,|^2$, where

$$\mathbf{r}(s,t) \;=\; \big(\mathbf{p}_0 \;+\; t\,\mathbf{u}\big) \;-\; \big(\mathbf{p}_1 \;+\; s\,\mathbf{v}\big)$$

has a minimum value and to find that value.

As noted above, we shall begin by proving that there is a minimum value. If we write out the conditions for a point to satisfy $\nabla f(s,t) = \mathbf{0}$ we obtain the following system of linear equations, where A and B are some constants.

$$t\,\langle \mathbf{u}, \mathbf{u}\rangle \;-\; s\,\langle \mathbf{u}, \mathbf{v}\rangle \;=\; A$$

$$t\,\langle \mathbf{u}, \mathbf{v}\rangle \;-\; s\,\langle \mathbf{v}, \mathbf{v}\rangle \;=\; B$$

These equations have a unique solution because the determinant

$$\begin{vmatrix} \langle \mathbf{u}, \mathbf{u}\rangle & \langle \mathbf{u}, \mathbf{v}\rangle \\ \langle \mathbf{u}, \mathbf{v}\rangle & \langle \mathbf{v}, \mathbf{v}\rangle \end{vmatrix}$$

is nonzero by the Schwarz inequality and the linear indepdendence of $\mathbf{u}$ and $\mathbf{v}$. Let $R > 0$ be so large that the solution $(s^*,\, t^*)$ lies inside the circle $s^2+t^2 = R^2$. Then on the set $s^2+t^2 \leq R^2$ either the minimum value occurs at the unique critical point or else it occurs on the boundary circle. Let D be the value of the function at the critical point, so that $D \geq 0$. If D is not a minimum value for $f(s,t)$ then for every $Q > R$ there is a point on the circle $s^2 + t^2 = Q^2$ for which the value of the function is less than D. We claim this is impossible, and it will follow that D must be the minimum value of the function.

Consider the values of the function f on the circle of radius ρ; these are given by

$$| \, \mathbf{r} \, (\, \rho \, \cos\theta, \; \rho \, \sin\theta \,) \, |^2$$

and if we write everything out explicitly we obtain the following expression for this function, in which $\mathbf{q}$ is the vector $\mathbf{p}_0 - \mathbf{p}_1$:

$$\rho^2 \, | \cos\theta \, \mathbf{u} - \sin\theta \, \mathbf{v}|^2 \;\; + \;\; 2 \, \rho \, \langle \cos\theta \mathbf{u} - \sin\theta \, \mathbf{v}, \; \mathbf{q} \rangle \;\; + \;\; |\mathbf{q}|^2$$

Let k denote the minimum value of $| \cos\theta \, \mathbf{u} - \sin\theta \, \mathbf{v}|$ for $\theta \in [0,\, 2\pi]$ and let K denote the maximum value. Since $\mathbf{u}$ and $\mathbf{v}$ are linearly independent, the displayed expression is always positive and therefore k must be positive. We claim that the minimum value of $f(s,t)$ on the circle $s^2+t^2 = \rho^2$ is greater than or equal to the following expression:

$$\rho^2 \, k^2 - 2 \, \rho \, K \, |\mathbf{q}| + |\mathbf{q}|^2$$

This follows immediately from the inequalities

$$\rho^2 \, | \cos\theta \, \mathbf{u} - \sin\theta \, \mathbf{v}|^2 \quad \geq \quad \rho^2 \, k^2$$

$$2 \, \rho \, \langle \cos\theta \mathbf{u} - \sin\theta \, \mathbf{v}, \; \mathbf{q} \rangle \quad \geq \quad - 2\rho \, | \cos\theta \, \mathbf{u} - \sin\theta \, \mathbf{v}| \cdot |\mathbf{q}| \quad \geq 2 \, \rho \, K \, |\mathbf{q}|$$

where the first inequality in the second line comes from the Schwarz inequality.

Since

$$\lim_{\rho \to \infty} \; \rho^2 \, k^2 - 2 \, \rho \, K \, |\mathbf{q}| + |\mathbf{q}|^2 \quad = \quad +\infty$$

it follows that all sufficiently large ρ the minimum value of $f(s,t)$ on the circle $s^2+t^2 = \rho^2$ is strictly greater than D, and therefore D must be the absolute minimum for f on the set $s^2 + t^2 \leq \rho^2$ for all sufficiently large ρ. But this means that D must be the absolute minimum for the function over all possible values of s and t.

SECOND STEP. In order to find a point where the minimum value is attained one needs to set the partial derivatives of f with respect to a and t both equal to zero. If we do this we obtain the following equations:

$$0 \quad = \quad 2 \, \mathbf{r}(s,t) \cdot (-\mathbf{v})$$

$$0 \quad = \quad 2 \, \mathbf{r}(s,t) \cdot (\; \mathbf{u})$$

Since $\mathbf{u}$ and $\mathbf{v}$ are linearly independent, this minimum occurs when $\mathbf{r}(s,t)$ a scalar multiple of $\mathbf{u} \times \mathbf{v}$. Suppose that the minimum is attained at parameter values (s_0, t_0). Then we have $\mathbf{r}(s_0, t_0) = k \, \mathbf{u} \times \mathbf{v}$ for some scalar k, and it follows immediately that the minimum distance d satisfies

$$d \quad = \quad \frac{| \; [\mathbf{u}, \mathbf{v}, \mathbf{r}(s_0, t_0)] \; |}{|\mathbf{u} \times \mathbf{v}|}$$

where $[\mathbf{a}, \mathbf{b}, \mathbf{c}]$ refers to the usual triple product of vectors having the form $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$. We claim that a similar formula holds with $\mathbf{r}(0,0) = \mathbf{p}_0 = \mathbf{p}_1$ replacing $\mathbf{r}(s_0, t_0)$. This is true because

$$\mathbf{r}(0,0) = \mathbf{r}(s_0, t_0) \ + \ t_0\,\mathbf{u} \ - \ s_0\,\mathbf{v_0}$$

which implies that the triple products $\big[\mathbf{u}, \mathbf{v}, \mathbf{r}(s_0, t_0)\big]$ and $\big[\mathbf{u}, \mathbf{v}, \mathbf{r}(0,0)\big]$ are equal. This proves the formula in its usual form

$$d \quad = \quad \frac{\big|\ \big[\mathbf{u}, \mathbf{v}, \mathbf{r}(0,0)\big]\ \big|}{|\mathbf{u} \times \mathbf{v}|}$$

if one assumes that the function $f(s,t)$ does attain a minimum value.

THIRD STEP. This follows fairly quickly from the argument in the second step. Let $\mathbf{x}_m$ and $\mathbf{y}_m$ be the points where $d(\mathbf{x}, \mathbf{y})$ is minimized, and write

$$\mathbf{x}_m \ + \ t_m\,\mathbf{u} \quad , \quad \mathbf{y}_m \ + \ s_m\,\mathbf{v}$$

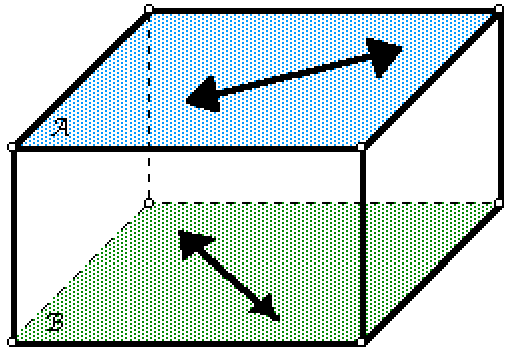
for suitable scalars t_m and s_m. Then $|\mathbf{r}(s_m, t_m)|$ and $|\mathbf{r}(s_m, t_m)|^2$ are the minimum values of the respective functions, and therefore by the first part of the proof for the second step we know that

$$0 \quad = \quad 2\,\mathbf{r}(s_m, t_m) \cdot (-\mathbf{v})$$

$$0 \quad = \quad 2\,\mathbf{r}(s_m, t_m) \cdot (\ \mathbf{u})$$

Since $\mathbf{r}(s_m, t_m) = \mathbf{x}_m - \mathbf{y}_m$, these equations imply that the line joining $\mathbf{x}_m$ and $\mathbf{y}_m$ is perpendicular to both L and M.■

Some pictures of skew lines

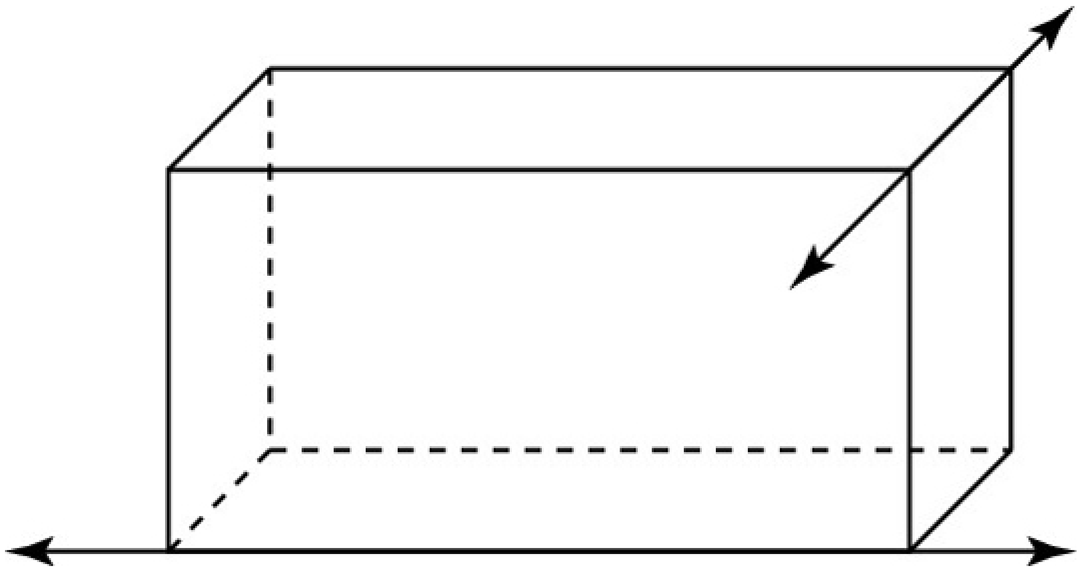


(Source: http://intermath.coe.uga.edu/dictnary/descript.asp?termID=424)

Think of the solid rectangular box as the set of all points (x, y, z) such that x lies between 0 and a, y lies between 0 and b, and z lies between 0 and c, where a, b and c are all positive real numbers. Then the line indicated by arrows in the bottom plane is the one joining the vertices $(0, b, 0)$ and $(a, 0, 0)$, while the line indicated by arrows in the top plane is the one joining the vertices $(0, 0, c)$ and (a, b, c). To see that the two lines are skew lines, it is enough to show that there is no plane containing these four vertices. But if such a plane existed then the points would satisfy an equation $Px+Qy+Rz = K$ for some P, Q, R, K where not all of the first three numbers are zero, and if we substitute the previous four points into this equation we get $Qb=K=Pa=Rc=Pa+Qb+Rc$. If this system has a solutions then adding the first three equations yields $K=Pa+Qb+Rc=3K$, which means that $K = 0$; this and the original system imply that a, b, c must all be zero. Since we are assuming all three numbers are positive, it follows that no plane contains the given four points.

Note that the two lines have a common perpendicular which is a vertical line through the center of the rectangular solid.

Yet another example of the same type appears on the next page.



(Source: http://www.dummies.com/how-to/content/getting-to-know-lines.html)

In this case the line on the lower face passes through $(0, 0, 0)$ and $(a, 0, 0)$, while the line on the upper face passes through $(a, 0, c)$ and (a, b, c). An argument like the preceding one shows that these four points are not coplanar. The details of checking this out are left as an exercise. In this case, the common perpendicular is the vertical line passing through $(a, 0, c)$ and $(a, 0, 0)$.