NOTE ON THE TRIANGLE CONGRUENCE AXIOMS

Our purpose here is to prove an assertion made at the beginning of Section II.4 of the notes; namely, the **SAS** triangle postulate implies the **ASA** and **SSS** postulates. Since this is an assertion about the logical interdependence of synthetic axioms, the proofs of these results must be entirely synthetic.

Proof that **SAS** implies **ASA**. Suppose that $|\angle ABC| = |\angle DEF|$, $d(B,C) = d(E,F)$, and $|\angle ACB| = |\angle DFE|$. If we can show that $d(A,B) = d(D,E)$, then by **SAS** we have $\Delta ABC \cong \Delta DEF$. We shall suppose that $d(A,B) \neq d(D,E)$ and derive a contradiction. There are two cases depending upon the direction of the inequality.

Suppose that $d(D,E) < d(A,B)$. There is a unique point $G \in (BA$ such that $d(B,G) = d(E,D)$, and since $d(B,G) = d(D,E) < d(B,A)$, we must have $B * G * A$. It follows that G lies in the interior of $\angle ACB$ and $\angle GBC = \angle ABC$. By **SAS** we have that $\Delta GBC \cong \Delta DEF$. This means that $|\angle GCB| = |\angle DFE|$. On the other hand, since G lies in the interior of $\angle ACB$ it also follows that $|\angle GCB| < |\angle ACB|$. Since we are assuming that $|\angle ACB| = |\angle DFE|$, the preceding sentences combine to show that

$$|\angle DFE| \quad = \quad |\angle GCB| \quad < \quad |\angle ACB| \quad = \quad |\angle DFE|$$

which is a contradiction. This means that we cannot have $d(D,E) < d(A,B)$.

Suppose that $d(D,E) > d(A,B)$. We can rule this out by interchanging the roles of A and D, B and E, and C and F in the preceding paragraph.

The preceding paragraphs show eliminate the possibility that $d(D,E) \neq d(A,B)$, so we must have $d(A,B) = d(D,E)$, and as noted above this suffices to prove that $\Delta ABC \cong \Delta DEF$.■

In order to prove **SSS**, we shall need some preparation. Note first that the proof of the Isosceles Triangle Theorem in the notes only uses **SAS** and **ASA**, and since the first of these implies the second, it follows that the Isosceles Triangle Theorem is a logical consequence of **SAS**, and as such we may use it in our derivation of **SSS** from **SAS**. Of course, we may also use **ASA** since we have derived this result from **SAS**. Next, we observe that the proofs of Proposition III.1.1 and Corollary III.1.2 in the notes are entirely synthetic and do not depend upon the postulates regarding congruent triangles, and therefore we may also use them in proving that **SAS** implies **SSS**. On the other hand, we do need to give a synthetic proof for Proposition III.1.7, whose proof in the notes involves the use of vectors.

PROPOSITION. *Suppose that we are given points $A \neq G$; let M be the midpoint of $[AG]$, and let B and C be distinct points not on the line AG such that $d(B,A) = d(B,G)$ and $d(C,A) = d(C,G)$. Then $M \in BC$ and BC is perpendicular to AG.*

Synthetic proof of the proposition. We shall first prove that BM is perpendicular to AG. By the Protractor Postulate, there is a ray $[BX$ such that $(BX$ lies on the same side of BA as G and $|\angle GBX| = \frac{1}{2}|\angle GBA|$. By the Trichotomy Principle we know that $(BX$ must be contained in the interior of $\angle GBA$, and therefore by the Additivity Postulate we have

$$|\angle ABX| = |\angle GBA| - \tfrac{1}{2}|\angle GBX| =$$
$$|\angle GBA| - \tfrac{1}{2}|\angle GBA| = \tfrac{1}{2}|\angle GBA| .$$

We can now apply the Crossbar Theorem to find a point N such that $N \in (BX \cap (AG)$. Since $d(B,A) = d(B,G)$, by **SAS** we have $\Delta ABN \cong \Delta GBN$. The latter implies two things: First, we have $d(A,N) = d(G,N)$, so that

$$d(A,G) = d(A,N) + d(A,N) = 2d(A,N) = 2d(G,N)$$

and hence $d(A,N) = \frac{1}{2}d(A,G) = \frac{1}{2}d(A,M)$; since there is only one point on (AG) satisfying the preceding equations, it follows that M and N must be the same point. Next, we have $|\angle BNA| = |\angle BNG|$. Since $A*N*G$ holds, it follows from the Supplement Postulate that

$$180^\circ = |\angle BNA| + |\angle BNG| = 2|\angle BNA| = 2|\angle BNG|$$

and therefore $|\angle BNA| = |\angle BNG| = 90^\circ$. Finally, since $M = N$, the preceding sentence implies that $BM \perp AG$.

If we replace B by C in the preceding paragraph, we obtain the analogous conclusion that $CM \perp AG$.

To complete the proof we need to show that $BM = CM$. There are two cases, depending upon whether B and C lie on the same or on opposite sides of AG. If they lie on the same side, then the Protractor Postulate implies that $[MB = [MC$, which in turn implies that $MB = MC$. On the other hand, if they lie on opposite sides and we choose Y such that $[MY = [MB^{\mathbf{OP}}$, then $(MY$ and $(MC$ lie on the same side of AG. By Proposition III.1 we know that $|\angle YMA| = 90^\circ$, and by the previous paragraph we also know that $|\angle CMA| = 90^\circ$, so by the Protractor Postulate we must have $[MC = [MY = [MB^{\mathbf{OP}}$. But this means that the lines MB and MC must be the same.■

Proof that **SAS** implies **SSS**. The idea again is to construct a congruent copy of ΔDEF which is nicely situated with respect to ΔABC and show that this copy must be ΔABC.

Specifically, first take the ray $(BW$ such that W and A lie on the same side of BC and $|\angle WBC| = |\angle DEF|$, and then take the point $G \in (BW$ such that $d(B,G) = d(E,D)$. We may now apply **SAS** to conclude that $\Delta GBC \cong \Delta DEF$. If $G = A$ then we have the conclusion to **SSS**, so suppose that $G \neq A$; we shall derive a contradiction from this.

Since $\Delta GBC \cong \Delta DEF$, it follows that $d(B,G) = d(E,D) = d(B,A)$ and similarly $d(C,G) = d(F,D) = d(A,C)$. By the proposition established above, this means that the midpoint M of $[AG]$ must lie on the line BC. On the other hand, since A and G lie on the same side of BC, the Plane Separation Postulate implies that the entire segment $[AG]$ lies on one side of BC, and thus we have a contradiction. The source of the contradiction is our assumption that $G \neq A$, so we must have $G = A$, and as noted above this implies **SSS**.■