CONCURRENCE AND 3–DIMENSIONAL GEOMETRY

The following theorem appears in many old textbooks on solid geometry.

THEOREM. *Let A, B, C be noncollinear points in space. Then the set of all points X such that $d(A, X) = d(B, X) = d(C, X)$ is the line which passes through the circumcenter of ΔABC and is perpendicular to the plane of A, B and C.*

Proof. If Q is the circumcenter of ΔABC, then Q is equidistant from all three of its vertices, so we know immediately that Q belongs to the set described in the statement of the theorem. Furthermore, we know that Q is the only point which belongs to both the set and the plane of A, B and C, for any such point must lie on the perpendicular bisectors of the segments $[AB]$, $[BC]$ and $[AC]$ and there is only one such point because the perpendicular bisectors are all distinct.

Suppose first that X lies on the line through Q which is perpendicular to the given plane and $X \neq Q$. Then we have $d(A, Q) = d(B, Q)$, $|\angle XQA| = 90° = |\angle XQB|$, and $d(X, Q) = d(X, Q)$, so that $\Delta XQA \cong \Delta XQB$ by **SAS**. Taking corresponding parts of these triangles, we find that $d(X, A) = d(X, B)$. Switching the roles of B and C in the preceding argument, we also see that $d(X, A) = d(X, C)$.

Conversely, suppose that X is equidistant from A, B and C. If $X = Q$ then we know that X belongs to the set described in the theorem, so suppose now that $X \neq Q$. It will be convenient to use vectors; if we square the defining equations for the set, we obtain the following:

$$|X - A|^2 \quad = \quad |X - B|^2 \quad = \quad |X - C|^2$$

If we expand the three expressions in this pair of equations and subtract $|X|^2$ from each expression, we obtain the equations

$$|A|^2 \; - \; 2\langle A, X\rangle \quad = \quad |B|^2 \; - \; 2\langle B, X\rangle \quad = \quad |C|^2 \; - \; 2\langle C, X\rangle \; .$$

Since Q is equidistant from A, B and C we have a similar pair of equations in which X is replaced by Q, and if we subtract the equations involving Q from the equations involving X we obtain a system of equations involving $X - Q$:

$$-2\langle A, Q - X\rangle \quad = \quad -2\langle B, Q - X\rangle \quad = \quad -2\langle C, Q - X\rangle$$

Obviously we can cancel the -2 factors, and if we subtract the first expression from the first and third we obtain yet another set of equations:

$$\langle B - A, Q - X\rangle \quad = \quad 0 \; , \qquad \langle C - A, Q - X\rangle \quad = \quad 0$$

Therefore the normal direction to the plane ABC is given by the line QX, and this means that QX must be perpendicular to that plane.■