SOME BASIC RESULTS IN NEUTRAL GEOMETRY

The purpose of this file is to provide details or references for proofs some results from Euclidean geometry which do not require the Fifth Postulate or equivalently Playfair's Postulate $(\mathbf{P-0})$ (*i.e.*, the setting called ***neutral geometry*** in the course notes). Specifically, the results under consideration are listed on pages $\mathbf{243-246}$ of the course notes. In some cases the proofs given earlier in the notes turn out to be valid in neutral geometry, and in a few other cases the proofs are given in the solutions to the exercises for Unit $\mathbf{V}$; for the sake of completeness we shall fill in all the remaining proofs. Following the discussion in Unit $\mathbf{V}$, we use the numbering of results from previous units of the notes.

Proposition II.2.4. *Suppose that* **A, B, C, D** *are four distinct collinear points satisfying the conditions* **A∗B∗D** *and* **B∗C∗D**. *Then* **A∗B∗C** *and* **A∗C∗D** *also hold.*

The proof which appears in the notes is also valid in neutral geometry. ■

Theorem II.2.5. *Let* **a, b, c** *be three distinct collinear points. Then either* **c ∈ (ab** *or else* **c ∈ (ab$^{\mathbf{OP}}$**. *In the first case we have*

[ab = [ac, [ab$^{\mathbf{OP}}$ = [ac$^{\mathbf{OP}}$, (ab = (ac, *and* **(ab$^{\mathbf{OP}}$ = (ac$^{\mathbf{OP}}$.**

In the second case we have

[ab = [ac$^{\mathbf{OP}}$, [ab$^{\mathbf{OP}}$ = [ac, (ab = (ac$^{\mathbf{OP}}$, *and* **(ab$^{\mathbf{OP}}$ = (ac.**

Proof. The proof which appears in the notes is **NOT** valid in neutral geometry because it is based upon vector geometry, so we need to give an entirely new argument, which must be synthetic.

First of all, since exactly one of **a, b, c** is between the other two, the definitions of the rays **(ab** and **(ab$^{\mathbf{OP}}$** implies that **c** belongs to exactly one of these open rays.

Suppose first that **c ∈ (ab.** Let **d** be a point such that **d∗a∗b,** so that **(ab = (ad$^{\mathbf{OP}}$, (ad = (ab$^{\mathbf{OP}}$, [ab = [ad$^{\mathbf{OP}}$** and **[ad = [ab$^{\mathbf{OP}}$.** The hypothesis on **c** implies that either **c = b** or **a∗c∗b** or **a∗b∗c;** since the conclusion is trivial in the first subcase, assume that one of the other two alternatives holds. In each of these cases, Proposition **II.2.4** implies that **d∗a∗c,** so that **(ac = (ad$^{\mathbf{OP}}$, (ad = (ac$^{\mathbf{OP}}$, [ac = [ad$^{\mathbf{OP}}$** and **[ad = [ac$^{\mathbf{OP}}$.**

Suppose now that **c ∈ (ab$^{\mathbf{OP}}$,** so that **c∗a∗b.** Then the definitions imply that **[ab = [ac$^{\mathbf{OP}}$, [ab$^{\mathbf{OP}}$ = [ac, (ab = (ac$^{\mathbf{OP}}$,** and **(ab$^{\mathbf{OP}}$ = (ac. ■**

Proposition II.2.8. *Let* **M** *denote either a line* **L** *in a plane* **P** *or a plane* **Q** *in space. Then the following hold:*

1. *If* **A** *and* **B** *are on the same side of* **M** *and* **B** *and* **C** *are on the same side of* **M,** *then* **A** *and* **C** *are on the same side of* **M.**
2. *If* **A** *and* **B** *are on the same side of* **M** *and* **B** *and* **C** *are on opposite sides of* **M,** *then* **A** *and* **C** *are on the same side of* **M.**

3. *If* **A** *and* **B** *are on opposite sides of* **M** *and* **B** *and* **C** *are on the same side of* **M,** *then* **A** *and* **C** *are on opposite sides of* **M.**

The proof which appears in the notes is also valid in neutral geometry. ■

Lemma II.2.10. *Let* **L** *be a line in the plane, and let* **M** *be a line in the plane which meets* **L** *in exactly one point. Then* **M** *contains points on both sides of* **L.**

The proof which appears in the notes is also valid in neutral geometry. ■

Proposition II.2.11. *Let* **L** *be a line in the plane, let* $\mathbf{H_1}$ *and* $\mathbf{H_2}$ *be the two half – planes determined by* **L,** *and let* **M** *be a line in the plane which meets* **L** *in exactly one point. Then each of the intersections* $\mathbf{H_1} \cap \mathbf{M}$ *and* $\mathbf{H_2} \cap \mathbf{M}$ *is an open ray.*

The proof which appears in the notes is also valid in neutral geometry. ■

Proposition II.2.12. *Let* **L** *be a line in the plane, let* **M** *be a line in the plane which meets* **L** *in exactly one point* **A,** *and let* **B** *and* **C** *be two other points on* **M.** *Then* **B** *and* **C** *lie on the same side of the line* **L** *if either* $\mathbf{A*C*B}$ *or* $\mathbf{A*B*C}$ *is true, and they lie on opposite sides of the line* **L** *if* $\mathbf{B*A*C}$ *is true.*

The proof which appears in the notes is also valid in neutral geometry. ■

Theorem II.2.13. (Pasch's "Postulate") *Suppose we are given* $\triangle\mathbf{ABC}$ *and a line* **L** *in the same plane as the triangle such that* **L** *meets the open side* **(AB)** *in exactly one point. Then either* **L** *passes through* **C** *or else* **L** *has a point in common with* **(AC)** *or* **(BC).**

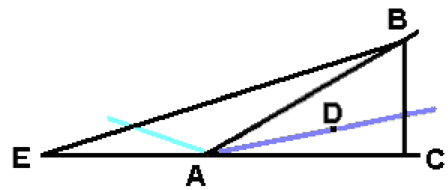
The proof which appears in the notes is also valid in neutral geometry. ■

Proposition II.3.1. *Let* **A** *and* **B** *be distinct points, and let* x *be a positive real number. Then there is a unique point* **Y** *on the open ray* **(AB** *such that* $d(\mathbf{A}, \mathbf{Y}) = x$*. Furthermore, we have* $\mathbf{A*Y*B}$ *if and only if* $x < d(\mathbf{A}, \mathbf{B})$*, and likewise we have* $\mathbf{A*B*Y}$ *if and only if* $x > d(\mathbf{A}, \mathbf{B})$.

Proof. This is worked out in Exercise **V.2.1**(a). ■

Theorem II.3.5. (Crossbar Theorem) *Let* **A, B, C** *be noncollinear points in* $\mathbb{R}^2$*, and let* **D** *be a point in the interior of* $\angle\mathbf{CAB}$*. Then the segment* **(BC)** *and the open ray* **(AD** *have a point in common.*

Proof. The proof which appears in the notes is **NOT** valid in neutral geometry because it is based upon vector geometry, so we need to give an entirely new argument, which must be synthetic.



Let **E** be a point such that $E*A*C$; observe that **E, B, C** are noncollinear, and **A** lies on both **AD** and **(EC).** Therefore by Pasch's "Postulate" the line **AD** must either pass through **B** or else contain a point from one of **(EB)** or **(BC);** we want to show the third alternative holds, so we have to eliminate the other two possibilities.

To see that **AD** does not pass through **B,** note that if it did then **A, D** and **B** would be collinear, and since **D** lies in the interior of $\angle$**CAB** this cannot happen.

We next need to show that **AD** does not meet **(EB);** assume to the contrary that they have some point **F** in common. By Proposition II.2.12 the betweenness relation $E*F*B$ implies that **F** and **B** lie on the same side of **AC;** we shall denote this open half – plane by $\mathcal{H}$. By Proposition II.2.11 $\mathcal{H} \cap$ **M** is an open ray, and since **D** lies on $\mathcal{H}$ this open ray must be **(AD.** Furthermore, since **F** also lies in the intersection, it follows that **F** lies on **(AD.** These observations in turn imply that **D** and **F** must lie on the same side $\mathcal{S}$ of **AB,** while $E*F*B$ implies that **E** also lies on $\mathcal{S}$ and since **D** lies in the interior of $\angle$**CAB** the same is true for **C;** combining these, we have shown that **C, D, E, F** all lie on $\mathcal{S}$. On the other hand, since $E*A*C$ is true by construction, Proposition II.2.12 implies that the points **E** and **C** must lie on opposite sides of **AB,** and hence we have reached a contradiction. The source of this contradiction is our assumption about the existence of the point **F,** and hence it follows that **AD** does not meet **(EB)** and consequently must meet the other open side, which is **(BC),** at some point **G.**

To conclude the proof, we must show that **G** also lies on **(AD.** Since **G** lies in **(BC)** it must belong to the interior of $\angle$**CAB,** so that **G** and **B** lie on the same side $\mathcal{H}$ of **AC,** and a final application of Proposition II.2.11 now shows that **G** must lie on the open ray **(AD.** ■

<u>Proposition II.3.6. (Trichotomy Principle)</u> *Let* **A** *and* **B** *be distinct points in* $\mathbb{R}^2$, *and let* **C** *and* **D** *be two points on the same side of* **AB**. *Then exactly one of the following is true*:

(1) **D** *lies on* **(BC** (*equivalently, the open rays* **(BC** *and* **(BD** *are equal*).

(2) **D** *lies in* Int $\angle$**ABC.**

(3) **C** *lies in* Int $\angle$**ABD.**

<u>Proof.</u> The proof which appears in the notes is **<u>NOT</u>** valid in neutral geometry because it is based upon vector geometry, so we need to give an entirely new argument, which must be synthetic.

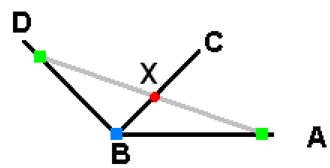
By the Plane Separation Postulate we know that exactly one of the following three statements is true:

(1) **D** *lies on* **BC.**

(2) **D** *lies on the same side of* **BC** *as* **A.**

(3) **D** *lies on the opposite side of* **BC** *as* **A.**

In the first case we can apply Proposition II.2.11 to conclude that **D** lies on **(BC,** and in the second case **D** lies in Int $\angle$**ABC** by the definition of the interior of an angle. Thus it is only necessary to prove that the third case implies the third alternative in the conclusion of the proposition. By the Plane Separation Postulate, in the third case we know that the line **BC** and the open segment **(AD)** have a point **X** in common.



Two applications of Proposition **II.2.12** now show that **X** lies in the interior of $\angle$**ABD.** Since **C** and **D** are assumed to lie on the same side $\mathcal{S}$ of **AB,** it follows that **X** must also lie on $\mathcal{S}$. Therefore by Proposition **II.2.11** the intersection of **BC** and $\mathcal{S}$ is an open ray containing both **C** and **X,** and this ray must also be contained in the side of **BC** containing **A.** In particular, it follows that **(BC = (BX** lies in the interior of $\angle$**ABD.** ■

Proposition II.3.7. (Vertical Angle Theorem) *Let* **A, B, C, D** *be four distinct points such that* **A**$*$**X**$*$**C** *and* **B**$*$**X**$*$**D.** *Then* $|\angle$**AXB**$|$ = $|\angle$**CXD**$|$.

The proof which appears in the notes is also valid in neutral geometry. ■

Theorem II.3.8. *Let* **A, B, C, D** *be distinct coplanar points, and suppose that* **C** *and* **D** *lie on the same side of* **AB.** *Then* $|\angle$**CAB**$|$ < $|\angle$**DAB**$|$ *is true if and only if* **C** *lies in the interior of* $\angle$**DAB.**

Theorem II.4.1. (Isosceles Triangle Theorem) *In* $\triangle$**ABC,** *one has* d(**A, B**) = d(**A, C**) *if and only if* $|\angle$**ABC**$|$ = $|\angle$**ACB**$|$.

The proof which appears in the notes is also valid in neutral geometry. ■

Corollary II.4.2. *In* $\triangle$**ABC,** *one has* d(**A, B**) = d(**A, C**) = d(**B, C**) *(the triangle is* ***equilateral****) if and only if one has* $|\angle$**ABC**$|$ = $|\angle$**ACB**$|$ = $|\angle$**BAC**$|$ *(the triangle is* ***equiangular****)*.

The proof which appears in the notes is also valid in neutral geometry. ■

Proposition III.1.1. *Let* **A, B, C** *be noncollinear points, and suppose that* **E** *is a point such that* **E**$*$**A**$*$**C** *holds. Then* **AB** $\perp$ **AC** *if and only if* $|\angle$**EAB**$|$ = $|\angle$**CAB**$|$.

The proof which appears in the notes is also valid in neutral geometry. ■

Corollary III.1.2. *Let* **A, B, C** *be noncollinear points, and suppose that* **D** *and* **E** *are points such that both* **E**$*$**A**$*$**C** *and* **B**$*$**A**$*$**D** *hold. Then* **AB** $\perp$ **AC** *if and only if*

$$|\angle \mathbf{CAB}| = |\angle \mathbf{EAB}| = |\angle \mathbf{EAD}| = |\angle \mathbf{DAC}| = 90^\circ.$$

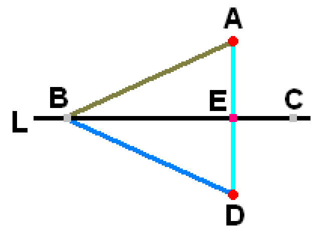
The proof which appears in the notes is also valid in neutral geometry. ■

Proposition III.1.3. *Let* **L** *be a line, let* **A** *be a point of* **L,** *and let* **P** *be a plane containing* **L.** *Then there is a unique line* **M** *in* **P** *such that* **A** $\in$ **M** *and* **L** $\perp$ **M.**

The proof which appears in the notes is also valid in neutral geometry. ■

Proposition III.1.4. *Let* **L** *be a line in the plane* **P,** *and let* **A** *be a point of* **P** ***not*** *on* **L.** *Then there is a unique line* **M** *such that* **A** $\in$ **M** *and* **L** $\perp$ **M.**

Proof. The proof which appears in the notes is **NOT** valid in neutral geometry because it is based upon vector geometry, so we need to give an entirely new argument, which must be synthetic.

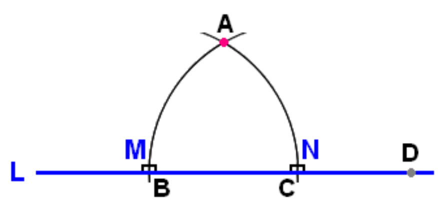


Existence of a perpendicular. Let **B** and **C** be two distinct points on **L**, and let **F** be a point on the side of **BC** opposite **A** such that $|\angle \mathbf{FBC}| = |\angle \mathbf{ABC}|$. Now let **D** be a point on **(BF** so that $d(\mathbf{D}, \mathbf{B}) = d(\mathbf{A}, \mathbf{B})$; since **A** and **D** lie on opposite sides of **L**, there is some point **E** where **(AD)** meets **L**. The proof splits into two cases, depending upon whether or not $\mathbf{B} = \mathbf{E}$.

If $\mathbf{B} = \mathbf{E}$ then we have $|\angle \mathbf{DEC}| = |\angle \mathbf{AEC}|$, and if we combine this with $\mathbf{A}*\mathbf{E}*\mathbf{D}$ then we find that the two angles are right angles and $\mathbf{L} = \mathbf{EC}$ is perpendicular to **AD**, proving the existence statement.

Suppose now that **B** and **E** are distinct points. Since **L** meets **AD** in **E**, it follows that **B** does not lie on $\mathbf{AD} = \mathbf{AE} = \mathbf{DE}$, which in turn implies that $\triangle \mathbf{ABE} \cong \triangle \mathbf{DBE}$ by **SAS.** The latter implies that $|\angle \mathbf{DEB}| = |\angle \mathbf{AEB}|$, and if we combine this with $\mathbf{A}*\mathbf{E}*\mathbf{D}$ then we find that the two angles are right angles and $\mathbf{L} = \mathbf{BE}$ is perpendicular to **AD.**

Uniqueness of perpendiculars. This portion of the proof relies upon Theorem **III.2.1** (the Exterior Angle Theorem), so we need to mention that the proof of the latter (given below) only depends upon results up to and including Corollary **III.1.2.**



Suppose that **M** and **N** are two perpendiculars to **L** (in the given plane) through the external point **A**, and let **B** and **C** be the points where **M** and **N** meet **L**. Let **D** be a point such that $\mathbf{B}*\mathbf{C}*\mathbf{D}$. Then the Exterior Angle Theorem implies that

$$90° = |\angle \mathbf{ACD}| > |\angle \mathbf{ABC}| = 90°$$

and hence we have reached a contradiction. The source of this contradiction is our assumption that there are two perpendiculars from **A** to **L**, and therefore there is only one perpendicular to **L** which passes through **A.** ■

Corollary III.1.5. *Suppose that* **L**, **M** *and* **N** *are three lines in the plane* **P** *such that we have* $\mathbf{L} \perp \mathbf{M}$ *and* $\mathbf{M} \perp \mathbf{N}$. *Then we also have* $\mathbf{L} \parallel \mathbf{N}$.

The proof which appears in the notes is also valid in neutral geometry. ■

Proposition III.1.7. *Let* **A** *and* **B** *be distinct points, let* **P** *be a plane containing them, suppose that* **D** *is the midpoint of* **[AB],** *and let* **M** *be the unique perpendicular to* **AB** *at* **D** *in the plane* **P.** *Then a point* $\mathbf{X} \in \mathbf{P}$ *lies on* **M** *if and only if* $d(\mathbf{X}, \mathbf{A}) = d(\mathbf{X}, \mathbf{B})$.

Proof. Only one part of the proof which appears in the notes is valid in neutral geometry because it is based upon vector geometry; namely, the case where **X** does not lie on the line **AB.** Therefore we need to give an entirely new argument for the case in which **X** lies on **AB,** and the new approach must be purely synthetic.

By the Ruler Postulate there is a $1-1$ correspondence between the points of **AB** and the real numbers $\mathbb{R}$ such that if the points **X** and **Y** on **AB** correspond to the real numbers x and y respectively, then we have

$$d(\mathbf{X}, \mathbf{Y}) = |x - y|.$$

Choose a and b such that **A** and **B** correspond to a and b respectively. Then the condition on **X** in the proposition translates into $|x - a| = |x - b|$. If we square both sides of this equation and subtract x^2 from each side, we obtain the equation $a^2 - 2ax = b^2 - 2bx$, which can be rewritten in the form $a^2 - b^2 = 2x(a - b)$. Since **A** and **B** are assumed to be distinct, the numbers a and b are unequal and therefore we can solve the equation to conclude that $x = ½(a + b)$, which means that **X** must be the midpoint of **(AB).** Conversely, if **X** is the midpoint then x is given as in the preceding sentence and we can easily check that $|x - a| = |x - b|$.■

Lines and planes in space. As indicated in the notes, one can prove several results on this topic without using Playfair's Postulate $(\mathbf{P}-\mathbf{0})$ or vector geometry, and the list of examples include Theorems **III.1.8, III.1.9,** and **III.1.12.** Purely synthetic proofs of these results are given at the end of this document in a separate subsection; ***none of these results will be needed in subsequent discussions of plane geometry in this course***.

Theorem III.2.1. (Exterior Angle Theorem) *Suppose we are given triangle* **△ABC,** *and let* **D** *be a point such that* **B*C*D.** *Then* **|∠ACD|** *is greater than* **|∠ABC|** *and* **|∠BAC|.**

Proof. The proof which appears in the notes is also valid in neutral geometry. Since the Exterior Angle Theorem is used in the neutral – geometric proofs of some results from Section **III.1,** we should mention that the proof of the Exterior Angle Theorem in the notes does not use anything from Section **III.1** of the course notes.■

Corollary III.2.2. *If* **△ABC** *is an arbitrary triangle, then the sum of any two of the angle measures* **|∠ABC|, |∠BCA|** *and* **|∠CAB|** *is less than* **180°.** *Furthermore, at least two of these angle measures must be less than* **90°.**

The proof which appears in the notes is also valid in neutral geometry.■

Corollary III.2.3. *Suppose we are given triangle* **△ABC,** *and assume that the two angle measures* **|∠BCA|** *and* **|∠CAB|** *are less than* **90°.** *Let* $\mathbf{D} \in \mathbf{AC}$ *be such that* **BD** *is perpendicular to* **AC.** *Then* **D** *lies on the open segment* **(AC).**

The proof which appears in the notes is also valid in neutral geometry.■

Corollary III.2.4. *Suppose we are given triangle* $\triangle$**ABC.** *Then at least one of the following three statements is true*:

(1) *The perpendicular from* **A** *to* **BC** *meets the latter in* **(BC).**
(2) *The perpendicular from* **B** *to* **CA** *meets the latter in* **(CA).**
(3) *The perpendicular from* **C** *to* **AB** *meets the latter in* **(AB).**

The proof which appears in the notes is also valid in neutral geometry. ■

Theorem III.2.5. *Given a triangle* $\triangle$**ABC,** *we have* $d(\mathbf{A}, \mathbf{C}) > d(\mathbf{A}, \mathbf{B})$ *if and only if we have* $|\angle\mathbf{ABC}| > |\angle\mathbf{ACB}|$.

The proof which appears in the notes is also valid in neutral geometry. ■

Theorem III.2.6. (Classical Triangle Inequality) *In* $\triangle$**ABC,** *we have the inequality* $d(\mathbf{A}, \mathbf{C}) < d(\mathbf{A}, \mathbf{B}) + d(\mathbf{B}, \mathbf{C})$.

Proof. This is worked out in Exercise **V.2.1(*c*).** ■

Proposition III.2.10. (Half of the Alternate Interior Angle Theorem) *Suppose we are given the setting and notation as in the notes. If the measures of one pair of alternate interior angles are equal, then the lines* **L** *and* **M** *are parallel.*

The proof which appears in the notes is also valid in neutral geometry. ■

Complement to the preceding result. *One obtains the same conclusion if the measures of one pair of alternate exterior angles are equal, or the measures of one pair of corresponding angles are equal.*

Sketch of proof. The reasoning in the proof of Corollary **III.2.12** shows that if the measures of one pair of alternate exterior angles or corresponding angles are equal, then the measures of one pair of alternate interior angles are equal, and hence one can apply Proposition **III.2.10** to conclude that the lines **L** and **M** are parallel. ■

Special case of the preceding results. *If two lines* **L** and **M** *are perpendicular to a third line, then they are parallel.* ■

Corollary III.2.15. (AAS Triangle Congruence Theorem) *Suppose we have two ordered triples of noncollinear points* **(A, B, C)** *and* **(D, E, F)** *satisfying* $d(\mathbf{B}, \mathbf{C}) = d(\mathbf{E}, \mathbf{F})$, $|\angle\mathbf{ABC}| = |\angle\mathbf{DEF}|$, *and* $|\angle\mathbf{CAB}| = |\angle\mathbf{FDE}|$. *Then we have* $\triangle\mathbf{ABC} \cong \triangle\mathbf{DEF}$.

Proof. This is worked out in Exercise **V.2.1(*d*).** ■

This congruence theorem turns out to be particularly important if we do not assume the Fifth Postulate; for example, see the neutral – geometric proof of the Hypotenuse – Side Congruence Theorem (Proposition V.2.1 in the notes).

Proposition III.3.1. *Suppose that* **A, B, C** *and* **D** *form the vertices of a convex quadrilateral. Then the open diagonal segments* **(AC)** *and* **(BD)** *have a point in common.*

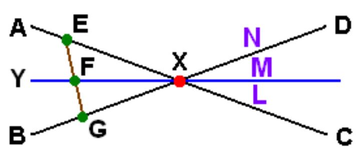
The proof which appears in the notes is also valid in neutral geometry. ■

We shall now show that some results in Section **III.1** on **3** – dimensional geometry can be proved by purely synthetic methods without using Playfair's Postulate**.**

Theorem III.1.8. *Suppose we are given a plane* **P** *and a line* **L** *not contained in* **P** *such that* **L** *and* **P** *meet at the point* **X.** *Suppose further that there are two distinct lines* **M** *and* **N** *in* **P** *such that* **X** *lies on both and* **L** *is perpendicular to both* **M** *and* **N.** *Then* **L** *is perpendicular to* **P.**

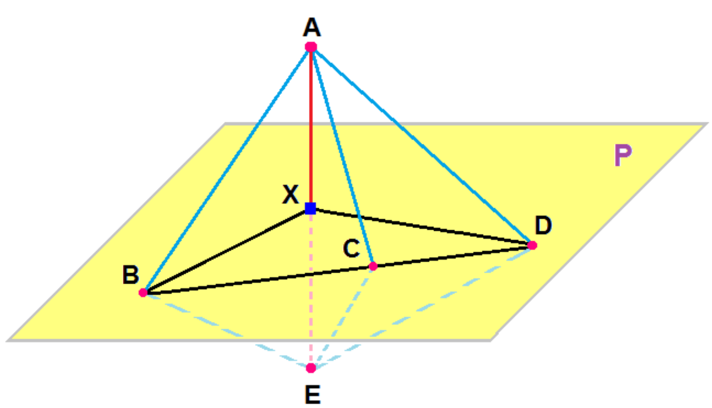
A complete synthetic proof of this theorem is fairly complicated**,** and we shall isolate two of the main steps as separate results**.**

Lemma III.1.8A. *Let* **P** *be a plane, let* **x** *is a point in* **P***, and let* **L,** **M** *and* **N** *be distinct lines passing through* **X.** Then there are points **E** ∈ **L,** **F** ∈ **M,** *and* **G** ∈ **M** *such that* **E**∗**F**∗**G.**



Proof. Let **A** and **C** be points of **L** such that **A**∗**X**∗**C,** and let **B** and **D** be points of **N** such that **B**∗**X**∗**D.** Since the three lines are distinct, the line **M** contains a point **Y** on the same side of **N** = **BD** as **A;** since the three lines are distinct and meet at **X,** it follows that **Y** does not lie on **L** = **XA.** Since **B** and **D** lie on opposite sides of **L** = **XA,** either the point **Y** lies in the interior of ∠**AXB** or else it lies in the interior of ∠**AXD.** Let **E** be a point of **(XA,** and let **G** be a point on **(XB** or **(XD,** depending upon whether **Y** lies in the interior of ∠**AXB** or ∠**AXD;** for these choices of points we have **Y** ∈ **Int**∠**EXG.** Therefore the Crossbar Theorem implies that there is a point **F** ∈ **(EG)** ∩ **(XY,** and the result follows because **M** = **XY.** ■

Lemma III.1.8B. *Let* **P** *be a plane, let* **A** *and* **E** *be points on opposite sides of* **P***, and suppose that* **B** *and* **D** *be points of* **P** *such that* d**(A, B)** = d**(E, B)** *and* d**(A, D)** = d**(E, D)***.* *If* **C** *is a point such that* **B**∗**C**∗**D,** *then* d**(A, C)** = d**(E, C)***.*



Proof. Note that the Space Separation Postulate implies there is a point $\mathbf{X} \in (\mathbf{AE}) \cap \mathbf{P}$. In a different direction, the hypotheses and SSS imply that $\triangle\mathbf{ABD} \cong \triangle\mathbf{EBD}$, so that $|\angle\mathbf{EBC}| = |\angle\mathbf{ABC}|$. The latter in turn implies that $\triangle\mathbf{ABC} \cong \triangle\mathbf{EBC}$ by SAS, which further implies that $d(\mathbf{A}, \mathbf{C}) = d(\mathbf{E}, \mathbf{C})$. ■

***Proof of Theorem* III.1.8.** Let **A** be a point of **L** different from **X**, and choose **E** such that **E** and **A** lies on opposite sides of **L**, with $\mathbf{A*X*E}$ and $d(\mathbf{A}, \mathbf{X}) = d(\mathbf{E}, \mathbf{X})$.

We need to show that if **T** is a line in **P** which passes through **X**, then **L** is perpendicular to **P**; since we know that **L** is perpendicular to **M** and **N**, we might as well assume that **T** is distinct from these two lines. Therefore Lemma III.1.8A implies there are points $\mathbf{B} \in \mathbf{M}$, $\mathbf{C} \in \mathbf{T}$, and $\mathbf{D} \in \mathbf{N}$ which are all distinct from **X** and satisfy $\mathbf{B*C*D}$.

By the hypotheses, both **XB** and **XD** lie in **P**, and both are perpendicular to **L** (at **X**), so that $|\angle\mathbf{AXB}| = |\angle\mathbf{EXB}| = |\angle\mathbf{AXD}| = |\angle\mathbf{EXD}| = 90°$. Therefore by SAS we have $\triangle\mathbf{AXB} \cong \triangle\mathbf{EXB}$ and $\triangle\mathbf{AXD} \cong \triangle\mathbf{EXD}$, which in turn imply $d(\mathbf{A}, \mathbf{B}) = d(\mathbf{E}, \mathbf{B})$ and $d(\mathbf{A}, \mathbf{D}) = d(\mathbf{E}, \mathbf{D})$. We can now apply Lemma III.1.8B to conclude that $d(\mathbf{A}, \mathbf{C}) = d(\mathbf{E}, \mathbf{C})$ also holds.

The equation in the preceding sentence and $d(\mathbf{A}, \mathbf{X}) = d(\mathbf{E}, \mathbf{X})$ imply $\triangle\mathbf{AXC} \cong \triangle\mathbf{EXC}$ by SAS, which in turn implies that $|\angle\mathbf{AXC}| = |\angle\mathbf{EXC}|$. The betweenness relation $\mathbf{A*X*E}$ and the Supplement Postulate for angle measures then imply that $|\angle\mathbf{AXC}| = |\angle\mathbf{EXC}| = 90°$, so that $\mathbf{L} = \mathbf{AX}$ is perpendicular to $\mathbf{T} = \mathbf{XC}$. Since **T** was arbitrary, this means that **L** must be perpendicular to the plane **P**. ■

We shall prove the next two results in the reverse of the order in which they appear in the notes.

<u>Theorem III.1.12.</u> *If* **L** *is a line and* **X** *is a point in space, then there is a unique plane through* **X** *which is perpendicular to* **L**.

Proof. There are two cases, depending upon whether or not **X** lies on **L**.

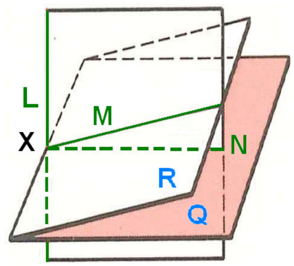
<u>Case III.1.12.1.</u> *Assume that* **X** *lies on* **L**.

<u>Existence of a perpendicular plane.</u> We first claim that there are two planes **Q** and **R** which intersect in the line **L**. The line **L** contains two points **A** and **B**, and there is some point **C** which does not lie on **L** because the latter is a proper subset of space. Let **Q** be the unique plane containing **A**, **B** and **C**; since **Q** is also a proper subset of space, there is some point **D** not on **Q**. Let **R** be the unique plane containing **A**, **B** and **D**. Then the planes **Q** and **R** are distinct because $\mathbf{D} \in \mathbf{R}$ but **D** does not belong to **Q**, and therefore the intersection of **Q** and **R**, which contains **A** and **B** and hence contains the entire line **L**, must be equal to $\mathbf{L} = \mathbf{AB}$.

Let **M** and **N** be lines in the planes **Q** and **R** which are perpendicular to **L** and pass through **X**. These two lines are distinct, for if $\mathbf{M} = \mathbf{N}$ then **Q** and **R** would both be planes which contain the same pair of intersecting lines and hence we would have $\mathbf{Q} = \mathbf{R}$. Next, let

P be the plane determined by the intersecting lines **M** and **N.** Then **L** is perpendicular to two lines in **P** through **X,** and therefore by Theorem **III.1.8** the line **L** and the plane **P,** which both pass through **X,** must be perpendicular. ■

Uniqueness of the perpendicular plane. Suppose that **Q** and **R** are two planes which contain **X,** and suppose that **L** is a line through **X** is perpendicular to both **Q** and **R.**



(Adapted from an illustration in A. M. Welchons, W. R. Krickenberger, and H. R. Pearson, ***Solid Geometry***, Ginn, Boston, 1959.)

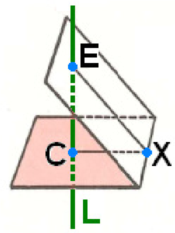
The planes **Q** and **R** have the point **X** in common, so they also have a line, say **K,** in common. Let **E** be a point on **Q** but not on **K,** let and **P** be the unique plane containing **L** and **E.** Then the intersection of **P** and **Q** is the line **XE = N;** since the intersection of **Q** and **R** is the line **K,** it follows that **N** is not contained in **R.** By construction **P** and **R** have the point **X** in common, so it follows that **P** and **R** intersect in some line **M;** this line is distinct from **M** because it is contained in **R** but **N** is not.

By hypothesis the line **L** is perpendicular to the planes **Q** and **R,** and therefore **L** is perpendicular to the lines **M** and **N.** All three of these lines pass through **X,** and by construction all three lie in **P.** Thus **M** and **N** are both perpendiculars to **L** at **X** in the plane P; however, this contradicts the uniqueness of perpendiculars in a plane, which was shown in Proposition **III.1.3.** The source of this contradiction was the assumption that there were two planes through **X** which were perpendicular to the line **L,** so there can be at most one such plane (and by the first half of the proof we know that there is at least one such plane). ■

Case III.1.12.2. *Assume that* **X** *does* **NOT** *lie on* **L.**

Existence of a perpendicular plane. Let **P** be the unique plane containing **L** and **X,** and let **M** be the unique line in **P** which passes through **X** and is perpendicular to **L.** As in the proof of the first case there is a second plane **Q** which contains **L.** Let **N** be the unique line in **Q** which passes through **X** and is perpendicular to **L.** Finally, let **R** be the unique plane containing the intersecting lines **M** and **N;** note that **R** contains **X** because **M** does. Since **L** is perpendicular to both **M** and **N,** Theorem **III.1.8** implies that **L** is perpendicular to **R.**

Uniqueness of the perpendicular plane. Suppose that **Q** and **R** are two planes which contain **X** such that **L** is perpendicular to both **Q** and **R.** Since **X** lies on both planes it follows that their intersection is a line which we shall call **K.** Let **C** and **E** denote the points at which **L** meets **Q** and **R.** The points **C** and **E** must be distinct, for otherwise **Q** and **R** would be two planes through **C = E** which are perpendicular to **L** at that common point, contradicting the first case of the theorem.



(Also adapted from Welchons – Krickenberger – Pearson, ***Solid Geometry.***)

By the perpendicularity hypotheses, both **EX** and **CX** are perpendicular to **L,** and by construction both lie in the unique plane **P** which contains **L** and **X.** Since there is a unique perpendicular to **L** through **X** in the plane **P,** it follows that **EX** = **CX,** and therefore the intersections of these lines with **L,** which are **E** and **C,** must be the same. This contradicts our previous conclusion about **C** and **E;** the source of this contradiction is our assumption that there are two perpendiculars to **L** through **X,** and hence there can be at most one perpendicular plane (and by the previous discussion there is at least one such plane).■

Before proceeding to the proof of Theorem **III.1.9** we shall derive one simple but important consequence of the previous theorem.

Corollary III.1.12A. *If* **L** *is a line in space*, **X** *is a point on* **L,** *and* **P** *is the plane perpendicular to* **L** *at* **X,** *then* **P** *contains every line* **M** *which passes through* **X** *and is perpendicular to* **L.**

Proof. Let **N** = **XY** be a line in **P** passing through **X,** so that **L** is perpendicular to **N;** if **M** and **N** are the same then the conclusion follows immediately, so assume that **M** and **N** are distinct. If **Q** is a plane containing the lines **M** and **N,** then Theorem **III.1.8** implies that **L** is perpendicular to **Q,** and therefore the uniqueness conclusion of Theorem **III.1.12** shows that **P** = **Q,** which means that **M** must be contained in **P.**■

Theorem III.1.9. *If* **P** *is a plane and* **X** *is a point in space, then there is a unique line through* **X** *which is perpendicular to* **P.**

Proof. There are again two cases, depending upon whether or not **X** lies on **P.**

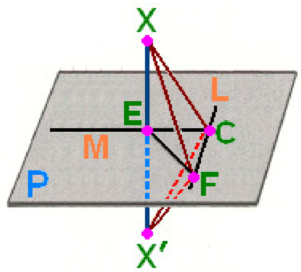
Case III.1.9.1. *Assume that* **X** *lies on* **P.**

Existence of a perpendicular line. Let **XA** are **XB** be two lines in **P** through **X.** By Theorem **III.1.12** there are planes **Q** and **R** through **X** such that **Q** is perpendicular to **XA** and **R** is perpendicular to **XB.** Since **X** is a common point of **Q** and **R,** the latter two planes intersect in some line **L.** By the definition of perpendicular planes, the line **L** is perpendicular to both **XA** and **XB.** These two lines lie in **P,** and therefore Theorem **III.1.8** implies that **L** is perpendicular to **P.**■

Uniqueness of the perpendicular line. Suppose that **L** and **M** are two perpendiculars to **P** at **X.** Let **Q** be the unique plane containing the intersecting lines **L** and **M.** Since **X** lies in both **P** and **Q,** their intersection is a line which we shall call **N.** The preceding statements imply that **L** and **M** are two lines in the plane **Q** which are perpendicular to **N,** and therefore Proposition **III.1.3** implies that **L** = **M,** a contradiction. Therefore there is at most one line perpendicular to **P** at **X** (and by the previous discussion there is at least one such line).■

<u>Case III.1.9.2.</u> *Assume that* **X** *does* **<u>NOT</u>** *lie on* **P.**

<u>Existence of a perpendicular line.</u> Since **P** contains at least three points, we know that there is some line **L** contained in **P.** If **Q** is the plane determined by **L** and **X,** then by Proposition **III.1.4** there is a unique line **XC** in **Q** such that **C** lies on **L** and **XC** is perpendicular to **L.** Let **M** be a line in P which is perpendicular to **L** and passes through **C** (as in the drawing on the next page).



(Also adapted from Welchons – Krickenberger – Pearson, ***Solid Geometry.***)

The point **X** does not belong to **M** because the latter is contained in **P** and the former is not, so by Proposition **III.1.4** there is a point **E** on **M** such that **EX** is perpendicular to **M.** Let **R** denote the plane containing **M** and **X.** There are now two cases depending upon whether or not **C** = **E** (in the drawing the two points are distinct).

If **C** = **E** then **XC** = **XE** is a line which is perpendicular to the lines **L** and **M** at this common point, and therefore by Theorem **III.1.8** the line **XE** is perpendicular to the plane determined by the intersecting lines **L** and **M;** since this plane is **P,** this proves the existence of a perpendicular line when **C** = **E.**

It remains to prove existence when **C** and **E** are distinct (as in the drawing). Let **F** be a point on **L** distinct from **C,** and choose **X′** so that **X**∗**E**∗**X′** and d(**E, X**) = d(**E, X′**). The two lines **M** and **CX** are contained in the plane R, and each is perpendicular to **L,** and therefore **L** is perpendicular to **R.** Since **X′** lies in **R** the line **X′C** also lies in **R,** and the conclusion in the preceding sentence implies that **X′C** is perpendicular to **L.** By **SAS** we have the right triangle congruence △**CEX** ≅ △**CEX′**, which in turn implies that d(**C, X**) = d(**C, X′**). Similarly, by **SAS** we have the right triangle congruence △**FCX** ≅ △**FCX′**, which implies that d(**F, X**) = d(**F, X′**). Finally, by **SSS** we have △**FEX** ≅ △**FEX′**, which implies that |∠**FEX**| = |∠**FEX′**|. Since we have by construction, the Supplement Postulate for angle measurement implies that |∠**FEX**| = |∠**FEX′**| = **90°.** This implies that **XX′** is perpendicular to **FE;** since **XX′** is also perpendicular to **CE** and the plane determined by the intersecting lines **CE** and **FE** is **P,** it follows that the line **XX′** is perpendicular to **P.** ■

<u>Note.</u> The proof given in the Welchons – Krickenberger – Pearson textbook overlooks the case where **C** = **E** (this is not a fatal error, but it emphasizes the need to be careful about checking whether two points are always distinct).

<u>Uniqueness of the perpendicular line.</u> Suppose that **L** and **M** are two perpendiculars to **P** which pass through **X.** Since **X** does not lie on **P** and the two lines are distinct, it follows that the points **A** and **B** at which **L** and **M** meet **P** must be distinct. If **N** is the line **AB** and **Q** is the unique plane containing **N** and **X**, then **L** and **M** are two lines in **Q** which pass through **X** and are perpendicular to **L.** This contradicts the uniqueness conclusion in Proposition **III.1.4**. The source of this contradiction is our assumption about two perpendiculars to **P** through **X**, and accordingly there is at most one line perpendicular to **P** at **X** (and by the previous discussion there is at least one such line). ■

General comments. It is immediately apparent that the proofs of the preceding $3-$ dimensional theorems in the notes, which use vector geometry, are much shorter and simpler than the synthetic proofs given here. One advantage of the latter is that they are also valid in neutral geometry; on the other hand, **the vector approach to perpendicular lines and planes in space provides a more unified approach to such properties in Euclidean geometry, with an added advantage that it extends fairly directly and clearly to more complicated situations in higher dimensions**.

Additional results about lines and planes in space

We shall conclude with synthetic proofs for a few other $3-$ dimensional results; as in the preceding discussion, all the proofs are valid in neutral geometry, and in at last one instance the argument is significantly more complex than its analytic/algebraic counterpart in the notes. The first theorem is the result on perpendicular bisectors in $3-$ dimensional space.

Theorem III.1.13. *Let* **A** *and* **B** *be distinct points in space. Then the set of all points that are equidistant from* **A** *and* **B** *is the plane which is perpendicular to the line* **AB** *and contains the midpoint of the segment* **[AB].**

Proof. Let **C** be the midpoint of **[AB].**

Suppose first that **X** is a point such that $d(\mathbf{X}, \mathbf{A}) = d(\mathbf{X}, \mathbf{B})$. The synthetic proof of the planar result (Proposition **III.1.7**) implies that a point **X** on **AB** satisfies the given equation if and only if $\mathbf{X} = \mathbf{C}$, so for the rest of this paragraph assume that **X** does not lie on **AB.** If **Q** is the plane determined by **AB** and **X**, then the synthetic proof of Proposition **III.1.7** shows that **X** lies on the perpendicular bisector **M(X; Q)** of **[AB]** in **Q.** Now **M(X; Q)** is a line through **C** which is perpendicular to **AB**, and therefore by Corollary **III.1.12A** the line **M(X; Q)** is contained in the plane **P** through **C** which is perpendicular to **AB.**

Conversely, suppose that **X** is a point which lies in the plane **P** through **C** such that **P** is perpendicular to **AB;** as before, we may as well assume that **X** does not lie on **AB.** If **Q** is the plane determined by **AB** and **X**, then the synthetic proof of the planar result (Proposition **III.1.7**) shows that $d(\mathbf{X}, \mathbf{A}) = d(\mathbf{X}, \mathbf{B})$. ■

We can also prove <u>**HALF**</u> of Theorem **III.1.11** in neutral geometry; namely, the statements which imply that two lines or planes are parallel. However, we shall not prove the other implications (if **U** and **V** are parallel and the line **T** is perpendicular to **U**, then **T** is also perpendicular to **V**); **in fact, it turns out that statements of this sort are equivalent to Playfair's Postulate.**

Theorem III.1.11N. *Let* **P** *and* **Q** *be distinct planes in space, and let* **L** *and* **M** *be distinct lines in space. Then the following hold*:

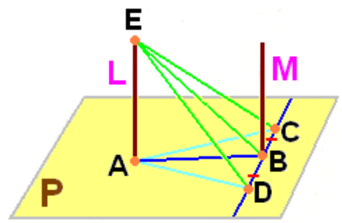
(1) *If both* **L** *and* **M** *are perpendicular to* **P,** *then* **L || M.**

(2) *If* **P** $\perp$ **L** *and* **Q** $\perp$ **L,** *then* **P || Q.**

Proofs. We shall consider the two parts of the theorem separately.

Proof of **(1).** Suppose that **P** meets **L** and **M** at **A** and **B** respectively. These two points are distinct, for otherwise **L** and **M** are both perpendicular to **P** at the same point and therefore **L = M.**

We need to prove that **L** and **M** are disjoint and coplanar, and we shall first prove that they are disjoint. If they had a common point **X,** then there would be some plane **S** containing both lines. Since **A** and **B** lie on **L** and **M,** both points lie in **S,** and within **S** the two lines **L** and **M** are perpendicular to **AB** which pass through **X.** This contradicts the uniqueness conclusion in Proposition **III.1.4,** and therefore it follows that **L** and **M** cannot have a common point.



By the preceding paragraph, it is only necessary to prove that **L** and **M** are coplanar. Let **BC** be a line in **P** which is perpendicular to **AB,** and choose **D** such that **C*B*D** and $d(\mathbf{B}, \mathbf{C}) = d(\mathbf{B}, \mathbf{D})$. By **SAS** we have the right triangle congruence $\triangle\mathbf{ABD} \cong \triangle\mathbf{ABC}$, which implies that $d(\mathbf{A}, \mathbf{C}) = d(\mathbf{A}, \mathbf{D})$. Let **E** be a point of **L** which is distinct from **A,** so that **EA** is perpendicular to **AC** and **AD** because **L** is perpendicular to **P.** Therefore by **SAS** we have the right triangle congruence $\triangle\mathbf{EAD} \cong \triangle\mathbf{EAC}$, which implies that $d(\mathbf{E}, \mathbf{C}) = d(\mathbf{E}, \mathbf{D})$.

The arguments in the preceding paragraphs show that **A** and **E** are equidistant from **C** and **D,** and therefore by Theorem **III.1.13** both **A** and **E** lie in the plane **Q** which is the perpendicular bisector of **[CD];** it follows that **L = AE** is also contained in **Q.** Also, since the line **M** is perpendicular to **CD** and passes through the midpoint **B** of **[CD],** Corollary **III.1.12A** implies that the line M is contained in this perpendicular plane Q. Thus we have shown that both L and M lie in the plane Q, and as noted above this completes the proof that L and M are parallel lines. ■

Proof of **(2).** Suppose that **L** meets **P** and **Q** at **A** and **B** respectively. These two points are distinct, for otherwise **P** and **Q** are both perpendicular to **L** at the same point and therefore **P = Q.**

Assume that **P** and **Q** have some common point **X.** We claim that **X** does not lie on **L;** if it did, then it would be the point at which **L** meets **P** and also the point at which **L** meets **Q,** so that **A = B,** which we know is false. Let **R** be the unique plane containing **L** and **X.** Then **XA** and **XB** are lines in **R** which are perpendicular to **L.** This contradicts the uniqueness conclusion in Proposition **III.1.4**, and the source of the contradiction is our assumption that there was a point **X** on both **P** and **Q.** Therefore no such point exists, which means that the planes must be parallel. ■