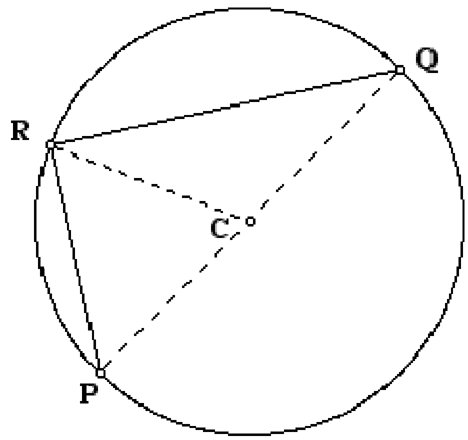
An angle inscribed in a semicircle is a right angle

This fact was known to the Babylonians and may have been one of the first results that Greek mathematicians proved using deductive logic (it is often attributed to Thales of Miletus, who probably lived from about 624 B.C.E. until about 548 B.C.E.).



(Source: http://mathforum.org/library/drmath/view/54662.html)

In this picture the angle is given by **QRP,** and **C** is the center of the circle**.** A proof of this result using vector geometry is given on the next page**.**

Proof of the theorem on right angles and semicircles

Here is a formal statement of the theorem:

THEOREM. *Suppose that $\angle ACB$ in the coordinate plane is inscribed in a* semicircle*; in other words, if X is the midpoint of the segment $[AB]$ then all three points A, B, C are equidistant from X. Then $\angle ACB$ is a right angle.*

Proof. We shall view the points in the coordinate plane as vectors and relabel them as $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, and $\mathbf{x}$. Since $\mathbf{x}$ is the midpoint of $\mathbf{a}$ and $\mathbf{b}$ it follows that $\mathbf{a} - \mathbf{x} = -(\mathbf{b} - \mathbf{x})$. Let

$$r \quad = \quad |\mathbf{a} - \mathbf{x}| \quad = \quad |\mathbf{b} - \mathbf{x}| \quad = \quad |\mathbf{c} - \mathbf{x}| \ .$$

In vector language, the conclusion of the theorem is that $\mathbf{a} - \mathbf{c}$ and $\mathbf{b} - \mathbf{c}$ are perpendicular, or equivalently that

$$(\mathbf{a} - \mathbf{c}) \cdot (\mathbf{b} - \mathbf{c}) \quad = \quad 0 \ .$$

Define new vectors

$$\mathbf{a}' \ = \ \mathbf{a} - \mathbf{x} \ , \quad \mathbf{b}' \ = \ \mathbf{b} - \mathbf{x} \ , \quad \mathbf{c}' \ = \ \mathbf{c} - \mathbf{x} \ .$$

It follows from the definitions that $\mathbf{a}' = -\mathbf{b}'$, and all three vectors $\mathbf{a}'$, $\mathbf{b}'$, $\mathbf{c}'$ have length r. Furthermore, we also have

$$\mathbf{a}' - \mathbf{c}' \quad = \quad \mathbf{a} - \mathbf{c} \ , \quad \mathbf{b}' - \mathbf{c}' \quad = \quad \mathbf{b} - \mathbf{c}$$

and therefore the conclusion of the theorem translates into the condition

$$(\mathbf{a}' - \mathbf{c}') \cdot (\mathbf{b}' - \mathbf{c}') \quad = \quad 0 \ .$$

Since $\mathbf{a}' = -\mathbf{b}'$, we may rewrite the expression on the left hand side as

$$(-\mathbf{b}' - \mathbf{c}') \cdot (\mathbf{b}' - \mathbf{c}') \quad = \quad -(\mathbf{b}' + \mathbf{c}') \cdot (\mathbf{b}' - \mathbf{c}') \quad = \quad -\big(|\mathbf{b}'|^2 - |\mathbf{c}'|^2 \big) \ .$$

Since $\mathbf{b}'$ and $\mathbf{c}'$ both have length r, it follows that this expression equals zero, which is what we needed to show in order to prove the theorem.■