III : Basic Euclidean concepts and theorems

The purpose of this unit is to develop the main results of Euclidean geometry using the approach presented in the previous units.

The choice of topics reflects the current subject requirements and recommendations for mathematics in the following State of California documents:

http://www.ctc.ca.gov/educator-prep/standards/SSMP-Handbook-Math.pdf

http://www.cde.ca.gov/ci/ma/cf/

We shall start by discussing perpendicularity and parallels, and we shall proceed to discuss standard material on triangles, quadrilaterals and regular polygons, the classical results on concurrence and similarity for triangles, some basic facts regarding intersections of a circle with a line or another circle, ending with a brief discussion of areas and volumes. References for further reading are also included.

III.1 : Perpendicular lines and planes

We shall follow the recommendation on page 36 (= online page 42) of the document http://www.ctc.ca.gov/educator-prep/standards/SSMP-Handbook-Math.pdf, which states, "*An introductory college geometry course should start from the beginning*." Much if not most of the material will be review, but one important new feature is that it discusses familiar elementary topics from the more advanced viewpoint of this course.

Perpendicular lines

We have already defined perpendicularity from the analytic approach in Section I.1; specifically, two intersecting lines $\mathbf{AB}$ and $\mathbf{AC}$ are ***perpendicular*** (written $\mathbf{AB} \perp \mathbf{AC}$) if and only if their inner product satisfies $(\mathbf{B} - \mathbf{A}) \cdot (\mathbf{C} - \mathbf{A}) = 0$. This is equivalent to the synthetic criterion $|\angle \mathbf{CAB}| = 90°$, and by the Supplement Postulate for angle measure we also have the following:

Proposition 1. *Let* $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ *be noncollinear points, and suppose that* $\mathbf{E}$ *is a point such that* $\mathbf{E}{*}\mathbf{A}{*}\mathbf{C}$ *holds. Then* $\mathbf{AB} \perp \mathbf{AC}$ *if and only if* $|\angle \mathbf{EAB}| = |\angle \mathbf{CAB}|$.

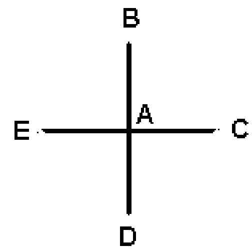
Proof. By the Supplement Postulate we have

$$|\angle \mathbf{EAB}| + |\angle \mathbf{CAB}| = 180°$$

and hence by elementary algebra we conclude that $|\angle \mathbf{EAB}| = |\angle \mathbf{CAB}|$ if and only if $2|\angle \mathbf{CAB}| = 180°$, which of course is equivalent to $|\angle \mathbf{CAB}| = 90°$.■

Corollary 2. *Let* **A, B, C** *be noncollinear points, and suppose that* **D** *and* **E** *are points such that both* **E*A*C** *and* **B*A*D** *hold. Then* **AB ⊥ AC** *if and only if*

$$|\angle \mathbf{CAB}| = |\angle \mathbf{EAB}| = |\angle \mathbf{EAD}| = |\angle \mathbf{DAC}| = 90°.$$



The corollary follows from repeated applications of the proposition.■

The Protractor Postulate and the preceding observations immediately yield the following result:

Proposition 3. *Let* **L** *be a line, let* **A** *be a point of* **L**, *and let* **P** *be a plane containing* **L.** *Then there is a unique line* **M** *in* **P** *such that* **A ∈ M** *and* **L ⊥ M.**

Note that the ***uniqueness only applies to lines in the given plane.*** In $3-$ dimensional space there are as many lines perpendicular to **L** at **A** as there are planes containing **L.** For example, if **L** is the usual $x-$ axis in $\mathbb{R}^3$, then a line **0C** through the origin is perpendicular to **L** if and only if the first coordinate of **C** is zero (and at least one of the other two coordinates is nonzero), which is equivalent to saying that the point **C** lies in the $yz-$ plane.

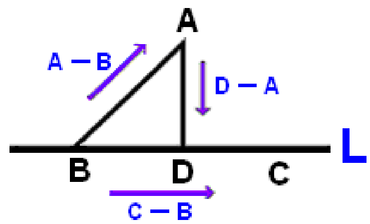
Proof. Let **B** be a second point on **L,** and **X** be a point of the plane **P** which is not on **L.** Then there is a unique ray **[AC** such that $|\angle \mathbf{CAB}| = 90°$ and **(AC** lies on the same side of **L** as **X.** It follows that **AC ⊥ AB** (where **AB = L**).

To prove uniqueness, suppose that **AD** is an arbitrary line in **P** such that **AD ⊥ AB** (where **AB = L**). There are two cases to consider, depending upon whether or not **C** and **D** lie on the same side of **L.** If they do, then by the uniqueness part of the Protractor Postulate we know that **[AD = [AC** and hence we also have that the lines **AD** and **AC** are identical. On the other hand, if **D** and **C** lie on opposite sides of **L,** take **E** to be a point such that **E*A*C.** Then **D** and **E** lie on the same side of **L,** so the uniqueness part of the Protractor Postulate now implies that **[AD = [AE,** which in turn implies **AD = AE.** Since **A, C** and **E** are distinct collinear points, the latter implies **AD = AC.**■

Of course, there is analogous result about perpendiculars if we are given a point **A** which does ***not*** lie on **L.**

Proposition 4. *Let* **L** *be a line, and let* **A** *be a point* ***not*** *on* **L.** *Then there is a unique line* **M** *such that* **A ∈ M** *and* **L ⊥ M.**

Proof. With the tools currently at our disposal, it is much easier and faster to do this analytically. Let **B** and **C** be distinct points of **L.** Express the vector **A − B** as a sum of the form **v + w,** where **v** is a scalar multiple of **C − B** and **w** is perpendicular to **C − B.** Set **D** equal to **v + B.**



We claim that **AD** is perpendicular to **L** and there is no other line **M** in the same plane such that **A** ∈ **M** and **M** ⊥ **L.** To see the first part, note that we have

$$\mathbf{A} - \mathbf{D} = (\mathbf{A} - \mathbf{B}) - (\mathbf{D} - \mathbf{B}) = (\mathbf{v} + \mathbf{w}) - \mathbf{v} = \mathbf{w}$$

and there is a (possibly zero) constant k such that $\mathbf{v} = (\mathbf{D} - \mathbf{B}) = k(\mathbf{C} - \mathbf{B})$. Therefore we have

$$(\mathbf{D} - \mathbf{A})\cdot(\mathbf{D} - \mathbf{B}) = \mathbf{w}\cdot[k(\mathbf{C} - \mathbf{B})] = k[\mathbf{w}\cdot(\mathbf{C} - \mathbf{B})] = k\cdot 0 = 0$$

so that **AD** is perpendicular to **L.**

It remains to show that there is only one perpendicular. Suppose that **E** ∈ **L** is such that **L** is perpendicular to **AE,** and write $\mathbf{E} - \mathbf{B} = x(\mathbf{C} - \mathbf{B})$ for a suitable scalar x. We then have

$$\mathbf{A} - \mathbf{E} = (\mathbf{A} - \mathbf{D}) - (\mathbf{E} - \mathbf{D}) = \mathbf{w} + (k - x)\cdot(\mathbf{C} - \mathbf{B})$$

so that

$$(\mathbf{A} - \mathbf{E})\cdot(\mathbf{C} - \mathbf{B}) = \big(\mathbf{w} + (k - x)\cdot(\mathbf{C} - \mathbf{B})\big)\cdot(\mathbf{C} - \mathbf{B}) = (k - x)\,||\,\mathbf{C} - \mathbf{B}\,||^2.$$

The lines **AE** and **L** are perpendicular if and only if this dot product vanishes, and since the length of **B − C** is positive, this can happen if and only if $k - x = 0$, which is equivalent to saying that **E = D.**■

Corollary 5. *Suppose that* **L, M** *and* **N** *are three lines in a plane* **P** *such that* **L ⊥ M** *and* **M ⊥ N.** *Then* **L || N.**

Proof. Take **B** and **C** to be the points where **M** meets **L** and **N** respectively. If **B = C,** then by uniqueness of perpendiculars at a point we would have **L = N;** since **L** and **N** are distinct, it follows that **B** and **C** are also distinct. If **L** and **N** were not parallel, then they would have a point **A** in common. This point could not lie on **M,** for if it did then it would be equal to both **B** and **C.** It would then follow that **L** and **N** would be distinct perpendiculars to **M** through the external point **A,** contradicting an earlier result. Therefore **L** and **N** cannot have any points in common, so that **L || N.**■

There is also a converse to the preceding corollary. We shall prove a more general result in the next section, but this special case is important enough in its own right to be mentioned separately.

Proposition 6. *Suppose that* **L, M** *and* **N** *are three lines in a plane* **P** *such that* **L || N** *and* **M ⊥ N.** *Then we also have* **L ⊥ M.**

Proof. We shall prove this result algebraically; express the plane **P** as **q + S,** where **S** is a $2-$dimensional vector subspace of $\mathbb{R}^3$. Similarly, write $\mathbf{L} = \mathbf{x}_0 + \mathbf{V}$ for some $1-$dimensional vector subspace **V,** and let $\mathbf{N} = \mathbf{z}_0 + \mathbf{V}$ where $\mathbf{z}_0$ does not lie on **L.** Let **v** be a nonzero vector in **V,** so that $\{\mathbf{v}\}$ forms a basis for **V.** Write $\mathbf{M} = \mathbf{w}_0 + \mathbf{U}$ for some $1-$dimensional subspace **U,** and let **u** be a nonzero vector in **U,** so that $\{\mathbf{u}\}$ forms a basis for **U.** Since all of the vectors $\mathbf{x}_0$, $\mathbf{z}_0$, $\mathbf{w}_0$ belong to **S,** it follows that

$$\mathbf{P} = \mathbf{x}_0 + \mathbf{S} = \mathbf{z}_0 + \mathbf{S} = \mathbf{w}_0 + \mathbf{S}$$

and since **L, M** and **N** a re all contained in **P** these imply that **U** and **V** are vector subspaces of **S.**

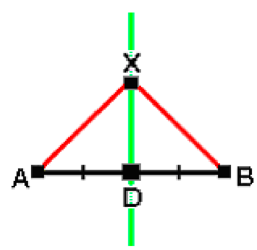
Since **M** and **N** are perpendicular, it follows that there is a point **q** which lies on both; it follows that **q + u** and **q + v** are second points of **M** and **N** respectively, and thus the perpendicularity condition on the lines means that $\mathbf{u} \cdot \mathbf{v} = 0$. Since these vectors belong to **S** and are nonzero, they are linearly independent and hence form a basis for **S.**

We next claim that **L** and **M** have a point in common; in other words, there are scalars a and b such that $\mathbf{x}_0 + a\mathbf{v} = \mathbf{w}_0 + b\mathbf{u}$. This follows because $\mathbf{x}_0 - \mathbf{w}_0$ lies in **S** and thus is a linear combination of **u** and **v.** Again, if **p** is this common point, then **p + u** and **p + v** are second points of **M** and **L** respectively, and since $\mathbf{u} \cdot \mathbf{v} = 0$ it follows that **M** and **L** are perpendicular.■

Perpendicular bisectors

We can now prove a result that is used very often in elementary geometry.

Proposition 7. (Planar Perpendicular Bisector Theorem) *Let* **A** *and* **B** *be distinct points, let* **P** *be a plane containing them, suppose that* **D** *is the midpoint of* **[AB],** *and let* **M** *be the unique perpendicular to* **AB** *at* **D** *in the plane* **P.** *Then a point* **X ∈ P** *lies on* **M** *if and only if* $d(\mathbf{X}, \mathbf{A}) = d(\mathbf{X}, \mathbf{B})$.



In classical language this is often stated as something like, "*The locus of points (in a plane) that are equidistant from two distinct points* **A** *and* **B** *is the* *perpendicular bisector* *of* **[AB].**"

TERMINOLOGY. This is a good time to mention that **the classical word *locus* in older geometry texts really has the same meaning as the modern word *set*.**

Proof. There are two cases depending upon whether or not **X** lies on **AB.** Suppose first that **X** lies on **AB.** Then $\mathbf{X} = \mathbf{A} + k(\mathbf{B} - \mathbf{A})$ for some scalar k, and we claim that k must be equal to ½ so that $\mathbf{X} = \mathbf{D}$. We may rewrite the expression for **X** equivalently as $\mathbf{X} = \mathbf{B} + (1-k)\cdot(\mathbf{A}-\mathbf{B})$, and therefore the equation $d(\mathbf{X}, \mathbf{A}) = d(\mathbf{X}, \mathbf{B})$, which is equivalent to the squared equation $d(\mathbf{X}, \mathbf{A})^2 = d(\mathbf{X}, \mathbf{B})^2$, is also equivalent to the following string of equations:

$$(1-k)^2\cdot||\mathbf{A}-\mathbf{B}||^2 = ||(1-k)\cdot(\mathbf{A}-\mathbf{B})||^2 = ||\mathbf{X}-\mathbf{B}||^2 =$$
$$||\mathbf{X}-\mathbf{A}||^2 = ||k\cdot(\mathbf{B}-\mathbf{A})||^2 = k^2\cdot||\mathbf{B}-\mathbf{A}||^2 = k^2\cdot||\mathbf{A}-\mathbf{B}||^2$$

Since the length of $\mathbf{A} - \mathbf{B}$ is positive, we may cancel it from the left and right sides to obtain the scalar equation $(1-k)^2 = k^2$, and the later reduces to $1 - 2k = 0$, so that $k = ½$ as claimed.

Suppose now that **X** does ***not*** lie on **AB.** If we have $\mathbf{XD} \perp \mathbf{AB}$ then by SAS we also have $\triangle\mathbf{XDA} \cong \triangle\mathbf{XDB}$, so that $d(\mathbf{X}, \mathbf{A}) = d(\mathbf{X}, \mathbf{B})$. Conversely, if the latter is true then we have $\triangle\mathbf{XDA} \cong \triangle\mathbf{XDB}$ by SSS, so that $|\angle\mathbf{XDA}| = |\angle\mathbf{XDB}|$. By previous results this means that $\mathbf{XD} \perp \mathbf{AB}$.■

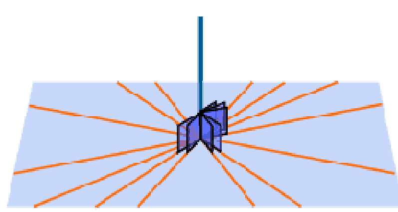
Perpendicularity and parallelism in space

> The ludicrous state of solid geometry … made me pass over this branch.
>
> Plato (428 B.C.E – 347 B.C.E.), ***The Republic***, Book VII

Three – dimensional geometry is considerably more complicated than its two – dimensional counterpart for many reasons, and accordingly it is not surprising that most accounts of elementary geometry only discuss solid geometry to a very limited extent. Many of the complications are already evident when one considers questions about parallel and perpendicular lines and planes in space, as we shall do in the final part of this section of the notes. Systematic use of linear algebra will simplify and clarify the discussion considerably.

The most basic notion involves perpendicularity of a line and plane in space.

Definition. Suppose that the line **L** and the plane **P** have a point **X** in common (but **L** is not contained in **P,** so there is only one such point). We shall say that ***the line* L *is perpendicular to the plane* P** and write $\mathbf{L} \perp \mathbf{P}$ if **L** is perpendicular to every line in **P** which passes through **X.**



(Source: http://www.mathsisfun.com/geometry/parallel-perpendicular-lines-planes.html)

It is easy to construct examples of lines which do not lie on the plane and are perpendicular to just one line in the plane. For example, take **P** to be the $xy-$plane in $\mathbb{R}^3$ and let **X** be the origin, so that **L** has the form **0v** where **v** is some nonzero vector. Suppose that we choose **v** to have coordinates $(1, 1, 1)$. A typical line through the origin in **P** consists of all points having the form $(tp, tq, 0)$, where p and q are not both zero. However, the only line of this form that is perpendicular to **0v** is the line defined by the equation $y = -x$.

The algebraic interpretation of a perpendicular line and plane is simple. If the plane is given by the equation $\mathbf{a}\cdot\mathbf{z} = \mathbf{b}$ and the line and plane meet at the point **x**, then **L** is the unique line joining **x** and $\mathbf{x}+\mathbf{a}$. Conversely, if **L** has the form $\mathbf{x} + \mathbf{V}$, where **V** is a $1-$dimensional vector subspace and **x** lies in both **L** and **P**, then **P** is defined by the equation $\mathbf{a}\cdot\mathbf{z} = \mathbf{a}\cdot\mathbf{x}$, where **a** is a nonzero vector in **V**. Furthermore, if we write $\mathbf{P} = \mathbf{x} + \mathbf{W}$ for some $2-$dimensional subspace **W**, then **W** is the vector subspace of all vectors perpendicular to the vectors in **V**, and **V** is the set of vectors which are perpendicular to all vectors in **W**.

In contrast to the example in the paragraph following the definition, we have the following.

<u>**Theorem 8.**</u> *Suppose we are given a plane* **P** *and a line* **L** *not contained in* **P** *such that* **L** *and* **P** *meet at the point* **x**. *Suppose further that there are two distinct lines* **M** *and* **N** *in* **P** *such that* **x** *lies on both and* **L** *is perpendicular to both* **M** *and* **N**. *Then* **L** *is perpendicular to* **P**.

<u>***Proof.***</u> Write $\mathbf{L} = \mathbf{x} + \mathbf{V}$ where **V** is spanned by the nonzero vector **v**. Let **y** and **z** be points in **P** such that **xy** and **xz** are distinct lines with $\mathbf{xy} \perp \mathbf{L}$ and $\mathbf{xz} \perp \mathbf{L}$. It follows that the vectors $\mathbf{z}-\mathbf{x}$ and $\mathbf{y}-\mathbf{x}$ form a basis for **W**. Suppose now that **w** is an arbitrary vector in **P** not equal to **x**. Then we have $\mathbf{w}-\mathbf{x} \in \mathbf{W}$ and hence

$$\mathbf{w}-\mathbf{x} = a(\mathbf{y}-\mathbf{x}) + b(\mathbf{z}-\mathbf{x})$$

for suitable scalars a and b. In order to prove the theorem we must show that the original line $\mathbf{L} = \mathbf{x}(\mathbf{x}+\mathbf{v})$ is perpendicular to the line **xw**, or equivalently that $\mathbf{v}\cdot(\mathbf{w}-\mathbf{x}) = 0$. The hypotheses imply that $\mathbf{v}\cdot(\mathbf{y}-\mathbf{x}) = \mathbf{v}\cdot(\mathbf{z}-\mathbf{x}) = 0$, and therefore we have

$$\mathbf{v}\cdot(\mathbf{w}-\mathbf{x}) = \mathbf{v}\cdot\big(a(\mathbf{y}-\mathbf{x}) + b(\mathbf{z}-\mathbf{x})\big) = a\mathbf{v}\cdot(\mathbf{y}-\mathbf{x}) + b\cdot(\mathbf{z}-\mathbf{x}) =$$

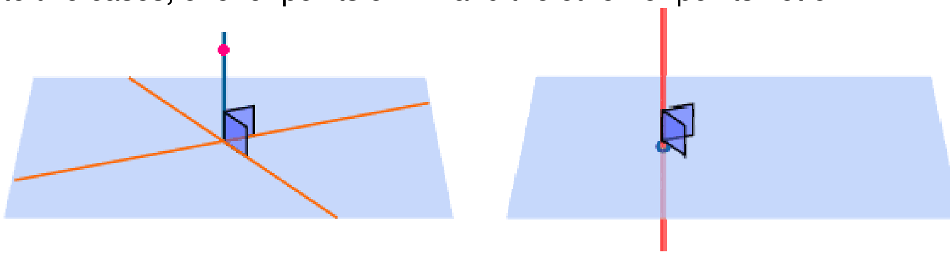
$$a\cdot 0 + b\cdot 0 = 0$$

which means that **L** is perpendicular to **xw**; since **w** was an arbitrary point of **P** not equal to **x**, it follows that $\mathbf{L} \perp \mathbf{P}$.■

There are some direct analogs to results in plane geometry.

<u>**Theorem 9.**</u> *If* **P** *is a plane and* **x** *is a point in space, then there is a unique line through* **x** *which is perpendicular to* **P**.

Note that we make no assumption whether or not **x** lies in **P,** and in fact the proof splits into two cases, one for points on **P** and the other for points not on **P.**



(Source: http://www.mathsisfun.com/geometry/parallel-perpendicular-lines-planes.html)

Proof. Write $\mathbf{P} = \mathbf{q} + \mathbf{W}$ for a suitable vector **q** and $2-$ dimensional vector subspace **W,** take a basis $\{\mathbf{u}_1, \mathbf{u}_2\}$ for **W,** and extend it to a basis for $\mathbb{R}^3$ by adding a single vector. Now apply the Gram – Schmidt process to obtain an orthonormal basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ such that the first two vectors form an orthonormal basis for **W.**

Suppose first that $\mathbf{x} \in \mathbf{P}$. Consider the line $\mathbf{L} = \mathbf{x}\mathbf{v}_3$; if **V** is the vector subspace spanned by $\mathbf{v}_3$, then **V** consists of all vectors perpendicular to **W** and vice versa, so by the by the algebraic description of perpendicular lines and planes we see that **L** is perpendicular to **P** at **x.** The preceding argument proves existence**.**

To prove uniqueness**,** suppose that $\mathbf{xy}$ is an arbitrary line that is perpendicular to **P.** Then $\mathbf{xy}$ is perpendicular to $\mathbf{x}\mathbf{v}_1$ and $\mathbf{x}\mathbf{v}_2$ in particular, so we conclude that $\mathbf{y} - \mathbf{x}$ is perpendicular to both $\mathbf{v}_1$ and $\mathbf{v}_2$. The only way a that linear combination $\mathbf{y} - \mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$ can satisfy this is if the coefficients of $\mathbf{v}_1$ and $\mathbf{v}_2$ are zero, which means that $\mathbf{y} - \mathbf{x}$ is a multiple of $\mathbf{v}_3$. Therefore **y** must lie in $\mathbf{x} + \mathbf{V} = \mathbf{L}$.

Suppose now that **x** does ***not*** lie in **P.** Let **z** be an arbitrary point of **P,** and expand the vector $\mathbf{x} - \mathbf{z}$ using the orthonormal basis in the first paragraph of the proof**:**

$$\mathbf{x} - \mathbf{z} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3$$

Let **u** be the sum of the first two terms of the displayed expression and let **w** be the third term**.** Since **x** does not lie in **P** we know that a_3 must be nonzero, and therefore it follows that **w** is nonzero**.** Set $\mathbf{x}_0 = \mathbf{z} + a_1\mathbf{v}_1 + a_2\mathbf{v}_2$, so that $\mathbf{x}_0 \in \mathbf{P}$, and consider the line $\mathbf{L} = \mathbf{x}_0\mathbf{y}$. Once again, the algebraic characterization of perpendicular lines and planes shows that **L** and **P** are perpendicular to each other, thus completing the proof of existence**.** Conversely, suppose now that we are given an arbitrary line **M** through **x** which is perpendicular to **P,** and let $\mathbf{w}_0$ be the point where this line **M** meets **P,** and let $\mathbf{w}_1 = \mathbf{x} - \mathbf{w}_0$. The perpendicularity condition implies that $\mathbf{w}_1$ is perpendicular to **W.** We then have

$$\mathbf{x} - \mathbf{z} = \mathbf{w}_0 + \mathbf{w}_1$$

where $\mathbf{w}_0$ lies in **W** and $\mathbf{w}_1$ is perpendicular to **W.** This in turn yields

$$\mathbf{x} - \mathbf{z} = \mathbf{w}_0 + \mathbf{w}_1 = (b_1\mathbf{v}_1 + b_2\mathbf{v}_2) + b_3\mathbf{v}_3$$

for suitably chosen scalars**.** By the uniqueness of expressions of a given vector in terms of a basis**,** the coefficients of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ in both these expressions must be equal. But this means that $\mathbf{w}_0 = \mathbf{x}_0$ and hence $\mathbf{w}_1 = \mathbf{w}$. Thus an arbitrary line through $\mathbf{x}$ which is perpendicular to $\mathbf{P}$ is equal to the line $\mathbf{L}$ constructed above, proving uniqueness.■

Following standard usage, we shall say that two planes $\mathbf{P}$ and $\mathbf{Q}$ in $\mathbb{R}^3$ are ***parallel*** if they have no points in common. We shall frequently write this as $\mathbf{P} \,||\, \mathbf{Q}$. Once again, the algebraic characterization of this is important.

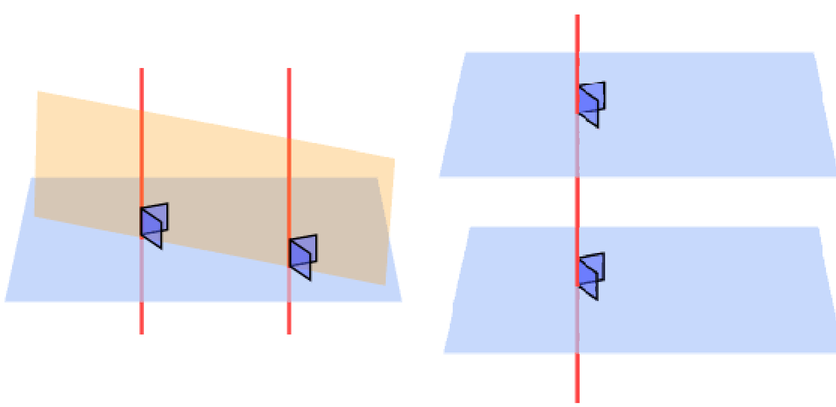
Lemma 10. *Let* $\mathbf{P}$ *and* $\mathbf{Q}$ *be distinct planes, and write* $\mathbf{P} = \mathbf{x} + \mathbf{W}$ *and* $\mathbf{Q} = \mathbf{z} + \mathbf{U}$ *for suitable* $2-$ *dimensional vector subspaces* $\mathbf{V}$ *and* $\mathbf{U}$ *respectively. Then* $\mathbf{P} \,||\, \mathbf{Q}$ *if and only if* $\mathbf{W} = \mathbf{U}$.

Proof. Suppose first that $\mathbf{P} \,||\, \mathbf{Q}$. If we translate this into a statement about linear equations, it means that we have a pair of nontrivial equations of the form $\mathbf{a} \cdot \mathbf{x} = b$ and $\mathbf{c} \cdot \mathbf{x} = d$ which have no simultaneous solution. By the basic results on solutions to systems of linear equations, this happens only if $\mathbf{a}$ and $\mathbf{c}$ are linearly dependent. In general, the solution spaces for the reduced equation $\mathbf{a} \cdot \mathbf{x} = 0$ and $\mathbf{c} \cdot \mathbf{x} = 0$ are merely the subspaces $\mathbf{W}$ and $\mathbf{U}$; if $\mathbf{a}$ and $\mathbf{c}$ are linearly dependent, then since they are both nonzero we know that each must be a nonzero scalar multiple of the other. But this means that $\mathbf{W} = \mathbf{U}$.

Conversely, suppose we are given distinct planes of the form $\mathbf{x} + \mathbf{W}$ and $\mathbf{y} + \mathbf{W}$. If they had some point $\mathbf{z}$ in common, then by the Coset Property from Section **I.3** we would have $\mathbf{x} + \mathbf{W} = \mathbf{z} + \mathbf{W} = \mathbf{y} + \mathbf{W}$, contradicting the fact that these planes are supposed to be distinct. Therefore we must have $\mathbf{x} + \mathbf{W} \,||\, \mathbf{y} + \mathbf{W}$.■

Theorem 11. *Let* $\mathbf{P}$ *and* $\mathbf{Q}$ *be distinct planes in space, and let* $\mathbf{L}$ *and* $\mathbf{M}$ *be distinct lines in space. Then the following hold*:

(1) *If both* $\mathbf{L}$ *and* $\mathbf{M}$ *are perpendicular to* $\mathbf{P}$, *then* $\mathbf{L} \,||\, \mathbf{M}$.
(2) *If* $\mathbf{L} \perp \mathbf{P}$ *and* $\mathbf{L} \,||\, \mathbf{M}$, *then* $\mathbf{M} \perp \mathbf{P}$.
(3) *If* $\mathbf{P} \perp \mathbf{L}$ *and* $\mathbf{Q} \perp \mathbf{L}$, *then* $\mathbf{P} \,||\, \mathbf{Q}$.
(4) *If* $\mathbf{L} \perp \mathbf{P}$ *and* $\mathbf{P} \,||\, \mathbf{Q}$, *then* $\mathbf{L} \perp \mathbf{Q}$.



(Source: http://www.mathsisfun.com/geometry/parallel-perpendicular-lines-planes.html)

Proofs. Let $\mathbf{V_1}$ and $\mathbf{V_2}$ be the $1-$dimensional vector subspaces corresponding to **L** and **M** respectively, and let $\mathbf{W_1}$ and $\mathbf{W_2}$ be the $2-$dimensional vector subspaces corresponding to **P** and **Q** respectively.

<u>Proof of **(1).**</u> In this case both $\mathbf{V_1}$ and $\mathbf{V_2}$ are the vector subspaces of all vectors perpendicular to $\mathbf{W_1}$; this implies that $\mathbf{V_1} = \mathbf{V_2}$, and hence that **L || M.**

<u>Proof of **(2).**</u> In this case $\mathbf{V_1} = \mathbf{V_2}$ and $\mathbf{V_1}$ is the vector subspace of all vectors perpendicular to $\mathbf{W_1}$. If the line **M** and the plane **P** have a point in common, this will imply that the line and plane are perpendicular, so we need only show that **M** and **P** have a point in common. Write $\mathbf{M} = \mathbf{x} + \mathbf{V_1}$ and $\mathbf{P} = \mathbf{y} + \mathbf{W_1}$. As in the preceding result, construct an orthonormal basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ such that the first two vectors form an orthonormal basis for $\mathbf{W_1}$. It will follow that the third vector gives a basis for $\mathbf{V_1}$. We then have

$$\mathbf{x} - \mathbf{y} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3$$

for appropriately chosen scalars a_1, a_2, a_3. It follows that

$$\mathbf{x} - a_3\mathbf{v}_3 = \mathbf{y} + a_1\mathbf{v}_1 + a_2\mathbf{v}_2$$

and since the left hand side lies in $\mathbf{V_1}$ and the right hand side lies in $\mathbf{W_1}$, we have found a vector belonging to both subsets. As noted before, this finishes the proof that $\mathbf{M} \perp \mathbf{P}$.

<u>Proof of **(3).**</u> Since **L** is perpendicular to both planes, it follows that $\mathbf{V_1}$ is the vector subspace of all vectors perpendicular to $\mathbf{W_1}$, and also $\mathbf{V_1} = \mathbf{V_2}$ is the vector subspace of all vectors perpendicular to $\mathbf{W_2}$. In particular, this means that $\mathbf{W_1}$ and $\mathbf{W_2}$ are both describable as the sets of vectors perpendicular to $\mathbf{V_1}$, which implies that $\mathbf{W_1} = \mathbf{W_2}$. Since **P** and **Q** are distinct, by the preceding lemma they must be parallel.

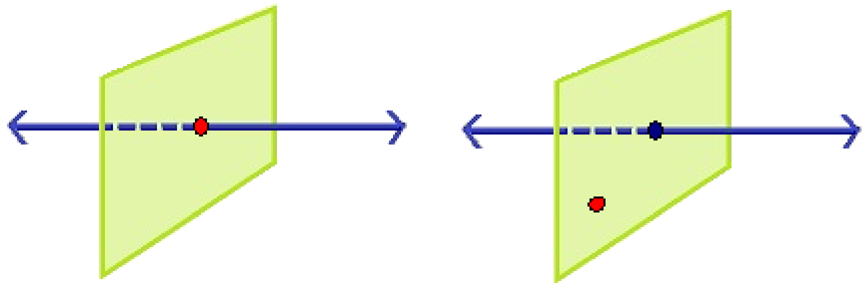
<u>Proof of **(4).**</u> In this case both $\mathbf{V_1}$ is the vector subspace of all vectors perpendicular to $\mathbf{W_1}$, and the latter is equal to $\mathbf{W_2}$. Thus $\mathbf{V_1}$ is also the vector subspace of all vectors perpendicular to $\mathbf{W_2}$, and since this perpendicular subspace is equal to $\mathbf{V_2}$ we must have $\mathbf{V_1} = \mathbf{V_2}$. As before we shall have $\mathbf{L} \perp \mathbf{Q}$ if we can show **L** and **Q** have a point in common. Write $\mathbf{L} = \mathbf{x} + \mathbf{V_2}$ and $\mathbf{P} = \mathbf{y} + \mathbf{W_2}$. Once again we have an orthonormal basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ such that the first two vectors form an orthonormal basis for $\mathbf{W_2}$. It will follow that the third vector gives a basis for $\mathbf{V_2}$. We then have

$$\mathbf{x} - \mathbf{y} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3$$

for appropriately chosen scalars a_1, a_2, a_3. Thus $\mathbf{x} - a_3\mathbf{v}_3 = \mathbf{y} + a_1\mathbf{v}_1 + a_2\mathbf{v}_2$ and since the left hand side lies in $\mathbf{V_2}$ and the right hand side lies in $\mathbf{W_2}$, we have a vector belonging to both subsets. As noted before, this finishes the proof that $\mathbf{L} \perp \mathbf{Q}$.■

The preceding result has a curious ***duality property:*** <u>*If we interchange the roles of lines and planes in the statements, we get the same conclusions in some rearranged order*</u>. Our next result is dual to the earlier one about dropping perpendiculars to a plane through a line.

Theorem 12. *If* **L** *is a line and* **x** *is a point in space, then there is a unique plane through* **x** *which is perpendicular to* **L.**



Note that ***we again make no assumption whether or not* x *lies in*** **L**, and in fact the proof again splits into two cases, one for points on **L** and the other for points not on **L.**

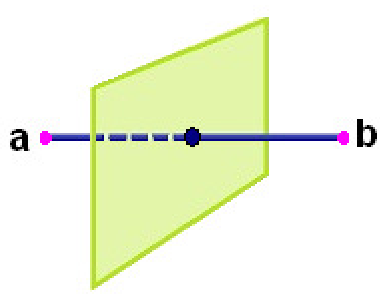
Proof. Start by writing **L** = **z** + **V** for some $1-$ dimensional vector subspace **V.** Once again we can extend $\{\mathbf{v}\}$ to a basis for $\mathbb{R}^3$, and in fact we can find an orthonormal basis $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ such that $\mathbf{w}_1$ is a positive multiple of **v.** Let **W** be the vector subspace spanned by the second and third vectors in the orthonormal basis.

Suppose first that $\mathbf{x} \in \mathbf{L}$. Then **L** may be rewritten as **x** + **V**, and the plane **x** + **W** will be perpendicular to **L**, proving existence. To verify uniqueness, let **x** + **U** be an arbitrary plane through **x** such that **x** is perpendicular to **L.** Then both **U** and **W** are the sets of all vectors perpendicular to **V**, and hence **W** = **U**; thus the perpendicular plane is unique in this case.

Suppose now that **x** does ***not*** lie on **L.** Then we have $\mathbf{z} - \mathbf{x} = a_1\mathbf{w}_1 + a_2\mathbf{w}_2 + a_3\mathbf{w}_3$ for suitable scalars a_1, a_2, a_3. We now have $\mathbf{z} - a_1\mathbf{w}_1 = \mathbf{x} + a_2\mathbf{w}_2 + a_3\mathbf{w}_3$ and if **y** is the point with these two equal descriptions, we see that **y** lies on **L**, it also lies on the plane **x** + **W**, and **L** is perpendicular to **x** + **W**, proving existence. To prove uniqueness, suppose that **Q** is a plane containing **x** such that $\mathbf{L} \perp \mathbf{Q}$. If **Q** is given by **x** + **U**, then both **W** and **U** consist of the vectors perpendicular to the span of $\mathbf{w}_3$, and therefore we must have **W** = **U.** This completes the argument when **x** does not lie on **L.**■

We could go much further in this direction, but we shall stop after one more result.

Theorem 13. *Let* **a** *and* **b** *be distinct points in space. Then the set of all points that are equidistant from* **a** *and* **b** *is the plane which is perpendicular to the line* **ab** *and contains their midpoint* ½(**a** + **b**).



In analogy with the planar case, the plane described in the theorem is called the ***perpendicular bisector (plane)*** of **a** and **b.**

<u>Proof.</u> We first write the equidistance equation in vector form $||\mathbf{x} - \mathbf{a}||^2 = ||\mathbf{x} - \mathbf{b}||^2$. Expanding this in the usual way we obtain

$$||\mathbf{a}||^2 - 2(\mathbf{a}\cdot\mathbf{x}) + ||\mathbf{x}||^2 = ||\mathbf{b}||^2 - 2(\mathbf{b}\cdot\mathbf{x}) + ||\mathbf{x}||^2$$

and if we subtract $||\mathbf{x}||^2$ from both sides and rearrange terms this becomes

$$||\mathbf{a}||^2 - 2(\mathbf{a}\cdot\mathbf{x}) = ||\mathbf{b}||^2 - 2(\mathbf{b}\cdot\mathbf{x}).$$

We may rewrite this further as $2(\mathbf{b} - \mathbf{a})\cdot\mathbf{x} = ||\mathbf{b}||^2 - ||\mathbf{a}||^2$. It is a simple exercise to compute that the midpoint $\frac{1}{2}(\mathbf{a} + \mathbf{b})$ satisfies this equation.

By the preceding paragraph, we know that the set of points equidistant from **a** and **b** is a plane containing $\frac{1}{2}(\mathbf{a} + \mathbf{b})$. Furthermore, the specific equation for **P** implies that if **L** is the line which is perpendicular to **P** at $\frac{1}{2}(\mathbf{a} + \mathbf{b})$, then $\mathbf{L} = \frac{1}{2}(\mathbf{a} + \mathbf{b}) + \mathbf{V}$, where **V** is the $1-$dimensional vector subspace spanned by $\mathbf{b} - \mathbf{a}$.

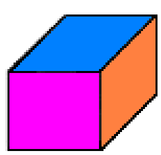
To conclude the proof, we need to verify that $\mathbf{L} = \mathbf{ab}$. In fact, direct computation yields

$$\mathbf{a} = \tfrac{1}{2}(\mathbf{a} + \mathbf{b}) - \tfrac{1}{2}(\mathbf{b} - \mathbf{a}) \in \tfrac{1}{2}(\mathbf{a} + \mathbf{b}) + \mathbf{V}$$

$$\mathbf{b} = \tfrac{1}{2}(\mathbf{a} + \mathbf{b}) + \tfrac{1}{2}(\mathbf{b} - \mathbf{a}) \in \tfrac{1}{2}(\mathbf{a} + \mathbf{b}) + \mathbf{V}$$

so that $\mathbf{a}, \mathbf{b} \in \mathbf{L}$ and hence $\mathbf{L} = \mathbf{ab}$.■

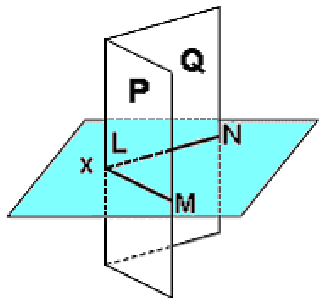
We shall conclude this section with a brief discussion of ***perpendicular planes,*** starting with a quick **<u>DEFINITION:</u>** Suppose **P** and **Q** are nonparallel planes in space defined by the nontrivial linear equations $\mathbf{a}\cdot\mathbf{x} = b$ and $\mathbf{c}\cdot\mathbf{x} = d$ respectively. Then **P** and **Q** are said to be ***perpendicular***, written $\mathbf{P} \perp \mathbf{Q}$, if and only if **a** and **c** are perpendicular. A rectangular box provides simple physical examples of perpendicular planes; at every corner there are three planes which meet, and each of them is perpendicular to the other two.



Before proceeding, we need to check that <u>*this definition does not depend upon the choices of equations defining the planes*</u>; in other words, if we are given (possibly) different equations $\mathbf{a}^*\cdot\mathbf{x} = b^*$ and $\mathbf{c}^*\cdot\mathbf{x} = d^*$, then $\mathbf{a}\cdot\mathbf{c} = 0$ if and only if $\mathbf{a}^*\cdot\mathbf{c}^* = 0$. To see this, observe that the only way two nontrivial linear equations can define the same plane is if one is obtained from the other by multiplying both sides by a nonzero scalar, so that we must have $\mathbf{a}^* = p\mathbf{a}$ and $b^* = pb$ for some nonzero constant p, and $\mathbf{c}^* = q\mathbf{a}$ and $d^* = qb$ for some nonzero constant q. Under these conditions it follows immediately that $\mathbf{a}\cdot\mathbf{c} = 0$ if and only if $\mathbf{a}^*\cdot\mathbf{c}^* = 0$.

The synthetic interpretation of perpendicular planes is given by the following result:

Theorem 14. *Suppose that* **P** *and* **Q** *are perpendicular planes in space, suppose that* **L** *is their line of intersection, and let* **x** *be a point on* **L.** *Then there are lines* **M** *and* **N** *through* **x** *such that* (1) $\mathbf{L} \perp \mathbf{M}$ *and* **M** *is contained in* **P**, (2) $\mathbf{L} \perp \mathbf{N}$ *and* **N** *is contained in* **Q**, (3) *we also have* $\mathbf{M} \perp \mathbf{N}$.



Corollary 15. *In the setting of the theorem we also have* $\mathbf{M} \perp \mathbf{Q}$ *and* $\mathbf{N} \perp \mathbf{P}$.

Proof of Corollary. By the theorem we know that **M** is perpendicular to two lines in **Q** through **x** and **N** is perpendicular to two lines in **P** through **x.**■

Proof of Theorem. Express the line **L** as $\mathbf{x} + \mathbf{U}$, where **U** is a $1-$dimensional vector subspace spanned by the nonzero vector **u.** Since $\mathbf{x} + \mathbf{u}$ lies on both **P** and **Q** we have

$$\mathbf{a} \cdot (\mathbf{x} + \mathbf{u}) \;=\; b \;=\; \mathbf{a} \cdot \mathbf{x} \quad \text{and} \quad \mathbf{c} \cdot (\mathbf{x} + \mathbf{u}) \;=\; d \;=\; \mathbf{c} \cdot \mathbf{x}$$

and hence $\mathbf{a} \cdot \mathbf{u} = \mathbf{c} \cdot \mathbf{u} = 0$, so that the vectors **a, c** and **u** are nonzero and mutually perpendicular. Let **M** be the line passing through **x** and $\mathbf{x} + \mathbf{c}$, and let **N** be the line passing through **x** and $\mathbf{x} + \mathbf{a}$. We then have

$$\mathbf{a} \cdot (\mathbf{x} + \mathbf{c}) \;=\; \mathbf{a} \cdot \mathbf{x} \;=\; b \quad \text{and} \quad \mathbf{c} \cdot (\mathbf{x} + \mathbf{a}) \;=\; \mathbf{c} \cdot \mathbf{x} \;=\; d$$

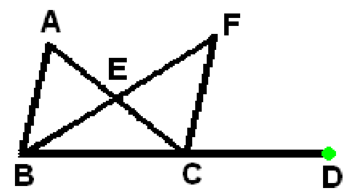
so that two points of **M** are contained in **P** (hence all of **M** is contained in **P**) and likewise two points of **N** are contained in **Q** (hence all of **N** is contained in **Q**). By construction we know that **L, M** and **N** are three lines which pass through **x** and any two of them are perpendicular to each other.■

III.2 : Basic results on triangles

One of the most important and best known results on a Euclidean triangle $\triangle \mathbf{ABC}$ is that the sum of the angle measurements $|\angle \mathbf{ABC}| + |\angle \mathbf{BCA}| + |\angle \mathbf{CAB}|$ is equal to 180 degrees. The goal of this section is to develop enough of the theory of triangles that we can prove this result.

The first result is often presented as a consequence of the result on the angle sum of a triangle, but for many reasons it is important in its own right. For example, the proof is valid for geometrical systems that do not necessarily satisfy Playfair's Postulate **P – 0.**

Theorem 1. (Exterior Angle Theorem) *Suppose we are given triangle* △**ABC,** *and let* **D** *be a point such that* **B**∗**C**∗**D.** *Then* |∠**ACD**| *is greater than both* |∠**ABC**| *and* |∠**BAC**|**.**



(***Source:*** **http://www.cut-the-knot.org/fta/Eat/EAT.shtml**)

Proof. Suppose we can show that |∠**ACD**| **>** |∠**BAC**|**.** Let **G** be a point such that **A**∗**C**∗**G.** Then by switching the roles of **A** and **B** and of **D** and **G,** we can also conclude that |∠**BCG**| **>** |∠**ABC**|**.** Since |∠**ACD**| **=** |∠**BCG**| by the Vertical Angle Theorem, it follows that |∠**ACD**| **>** |∠**ABC**|**.** Therefore it will suffice to prove the inequality |∠**ACD**| **>** |∠**BAC**|**.**

Let **E** be the midpoint of **[AC],** and let **F** ∈ **[EB**$^{\mathbf{OP}}$ be the unique point such that d(**E, F**) **=** d(**E, B**)**.** Then the midpoint condition implies d(**E, C**) **=** d(**E, A**)**,** and the Vertical Angle Theorem implies |∠**AEB**| **=** |∠**CEF**|**,** so that △**AEB** ≅ △**CEF** by **SAS.** It follows that |∠**BAE**| **=** |∠**ECF**|**.** Note that ∠**BAE** **=** ∠**BAC** and ∠**CFE** **=** ∠**ACF** by construction**.**

Since |∠**BAE**| **=** |∠**ACF**|**,** it will suffice to prove that |∠**ACF**| **<** |∠**ACD**|**,** and we shall have the latter if we can show that **F** lies in the interior of ∠**ACD.** The order relations **A**∗**E**∗**C** and **F**∗**E**∗**B** show that **A, E** and **F** all lie on the same side of the line **CD** **=** **BC.** Similarly, the order relations **B**∗**E**∗**F** and **B**∗**C**∗**D** show that **D** and **F** all lie on the same side of the line **EC** **=** **AC.** The preceding two sentences combine to show that **F** lies in the interior of ∠**ACD,** which by the previous observations implies the desired inequalities |∠**ACF**| **<** |∠**ACD**| and |∠**BAC**| **<** |∠**ACD**|**.**■

The preceding result has an extremely large number of important consequences**.** We limit ourselves here to some that will be needed repeatedly**.**

Corollary 2. *If* △**ABC** *is an arbitrary triangle, then the sum of any two of the angle measures* |∠**ABC**|**,** |∠**BCA**| *and* |∠**CAB**| *is less than* $180°$*. Furthermore, at least two of these angle measures must be less than* $90°$*.*

Proof. We use the notation of the preceding theorem**.** The argument for the latter and the Additivity and Supplement Postulates for angle measures show that

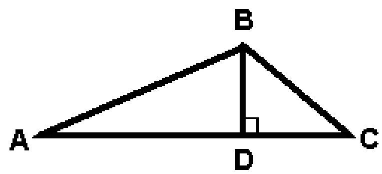
$$|\angle BCA| + |\angle CAB| = |\angle BCA| + |\angle ACF| = |\angle BCF| =$$

$$180^\circ - |\angle DCF| < 180^\circ.$$

The other two inequalities $|\angle CAB| + |\angle ABC| < 180^\circ$ and $|\angle ABC| + |\angle BCA| < 180^\circ$ follow from the same argument by interchanging the roles of **A, B** and **C.**

To prove the second statement, suppose that the measure of at least one of the vertex angles is at least 90°. Without loss of generality, we may assume that $|\angle ABC| \geq 90^\circ$; the other two cases can be shown similarly by permuting the roles of **A, B** and **C.** By the already proven first sentence in this corollary, we know that $|\angle CAB| + |\angle ABC| < 180^\circ$ and $|\angle ABC| + |\angle BCA| < 180^\circ$, so standard algebra implies that both of the angle measurements $|\angle CAB|$ and $|\angle BCA|$ must be less than 180°.■

Corollary 3. *Suppose we are given triangle* $\triangle$**ABC,** *and assume that the two angle measures* $|\angle BCA|$ *and* $|\angle CAB|$ *are less than* 90°. *Let* **D** ∈ **AC** *be such that* **BD** *is perpendicular to* **AC.** *Then* **D** *lies on the open segment* **(AC).**



Proof. We know that **D** cannot be equal to either **A** or **C,** because this would imply that either $|\angle BCA|$ or $|\angle CAB|$ would be equal to 90°. Thus one of the three points **A, C, D** must be between the other two. If we have **A**∗**C**∗**D,** then the Exterior Angle Theorem would imply that $|\angle ACB| > |\angle CDB| = 90^\circ$, which would contradict our assumption that $|\angle ACB| = |\angle BCA| < 90^\circ$. Similarly, if we have **D**∗**A**∗**C,** then the Exterior Angle Theorem would imply that $|\angle BAC| > |\angle BDA| = 90^\circ$, which would contradict our assumption that $|\angle CAB| = |\angle BAC| < 90^\circ$. The only remaining possibility for the collinear points **A, B, D** is the betweenness relation **A**∗**D**∗**C.**

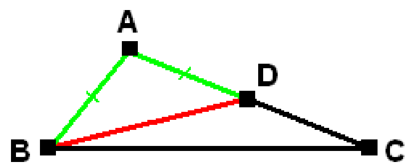
Corollary 4. *Suppose we are given triangle* $\triangle$**ABC.** *Then at least one of the following three statements is true*:

(1) *The perpendicular from* **A** *to* **BC** *meets the latter in* **(BC).**
(2) *The perpendicular from* **B** *to* **CA** *meets the latter in* **(CA).**
(3) *The perpendicular from* **C** *to* **AB** *meets the latter in* **(AB).**

This follows because the measures of at least two vertex angles are less than 90°.■

One can also use the Exterior Angle Theorem to prove the following ***complement to the Isosceles Triangle Theorem.***

Theorem 5. *Given a triangle* $\triangle$**ABC,** *we have* $d(\mathbf{A}, \mathbf{C}) > d(\mathbf{A}, \mathbf{B})$ *if and only if we have* $|\angle ABC| > |\angle ACB|$.



Less formally, this theorem states that ***the larger angle is opposite the longer side.***

<u>Proof.</u> Suppose that $d(\mathbf{A}, \mathbf{C}) > d(\mathbf{A}, \mathbf{B})$, and let **D** ∈ **(AC** be such that $d(\mathbf{A}, \mathbf{D}) = d(\mathbf{A}, \mathbf{B})$. Then $d(\mathbf{A}, \mathbf{D}) = d(\mathbf{A}, \mathbf{B}) < d(\mathbf{A}, \mathbf{C})$ implies that **D** lies on **(AC),** so that we have **A**∗**D**∗**C.** In particular, it also follows that **D** lies in the interior of ∠**ABC,** so that we have $|\angle \mathbf{ABC}| > |\angle \mathbf{ABD}|$. The Isosceles Triangle Theorem now implies that $|\angle \mathbf{ABD}| = |\angle \mathbf{ADB}|$, and the Exterior Angle Theorem implies that

$$|\angle \mathbf{ADB}| > |\angle \mathbf{DCB}| = |\angle \mathbf{ACB}|;$$

the final equation holds because the two angles are identical. If we string all these inequalities and equations together, we conclude that $|\angle \mathbf{ABC}| > |\angle \mathbf{ACB}|$.

Similarly, if we have $d(\mathbf{A}, \mathbf{C}) < d(\mathbf{A}, \mathbf{B})$, then by interchanging the roles of **B** and **C** in the preceding argument we can conclude that that $|\angle \mathbf{ABC}| < |\angle \mathbf{ACB}|$.

Suppose now that we have the converse situation with that $|\angle \mathbf{ABC}| > |\angle \mathbf{ACB}|$. If $d(\mathbf{A}, \mathbf{C}) = d(\mathbf{A}, \mathbf{B})$, then by the Isosceles Triangle Theorem we obtain the contradictory conclusion $|\angle \mathbf{ABC}| = |\angle \mathbf{ACB}|$. Likewise, if $d(\mathbf{A}, \mathbf{C}) < d(\mathbf{A}, \mathbf{B})$, then by the preceding paragraph we have $|\angle \mathbf{ABC}| < |\angle \mathbf{ACB}|$, which again contradicts our assumption. Therefore $d(\mathbf{A}, \mathbf{C}) > d(\mathbf{A}, \mathbf{B})$ is the only alternative consistent with the condition $|\angle \mathbf{ABC}| > |\angle \mathbf{ACB}|$.■

Some algebraic proofs

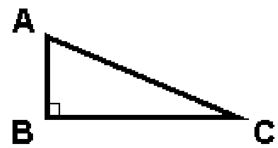
Up to this point we have used synthetic methods to prove our results. However, there are also some results which a more easily proved using algebraic methods, and before proceeding to the goal of this section we shall present them.

<u>Theorem 6. (Classical Triangle Inequality)</u> *Given* △**ABC,** *we have the inequality* $d(\mathbf{A}, \mathbf{C}) < d(\mathbf{A}, \mathbf{B}) + d(\mathbf{B}, \mathbf{C})$.

<u>Proof.</u> By the version of the Triangle Inequality in Section **I.1,** we know that the left hand side is less than or equal to the right hand side, and equality holds only if **A, B** and **C** are collinear. Since they are not, we must have strict inequality in this situation.■

The next result is generally regarded as one of the most important in all of Euclidean geometry.

<u>Theorem 7. (Pythagorean Theorem)</u> *If* △**ABC** *has a right angle at* **B,** *so that* **AB** ⊥ **BC,** *then* $d(\mathbf{A}, \mathbf{C})^2 = d(\mathbf{A}, \mathbf{B})^2 + d(\mathbf{B}, \mathbf{C})^2$.



Proof. We know that $d(\mathbf{A}, \mathbf{C})^2 = ||\mathbf{C} - \mathbf{A}||^2$, and since

$$(\mathbf{C} - \mathbf{A}) = (\mathbf{C} - \mathbf{B}) + (\mathbf{B} - \mathbf{A})$$

the expression $||\mathbf{C} - \mathbf{A}||^2$ is equal to

$$||\mathbf{C} - \mathbf{B}||^2 + 2(\mathbf{C} - \mathbf{B}) \cdot (\mathbf{B} - \mathbf{A}) + ||\mathbf{B} - \mathbf{A}||^2.$$

Since $\mathbf{AB} \perp \mathbf{BC}$, we know that $(\mathbf{C} - \mathbf{B}) \cdot (\mathbf{B} - \mathbf{A}) = 0$, and therefore the right hand side reduces to $||\mathbf{C} - \mathbf{B}||^2 + ||\mathbf{B} - \mathbf{A}||^2 = d(\mathbf{A}, \mathbf{B})^2 + d(\mathbf{B}, \mathbf{C})^2$, as required.■

In fact, the argument above yields the following stronger conclusion:

<u>**Theorem 8.** (Law of Cosines)</u> *Given* $\triangle\mathbf{ABC}$, *we have*

$$d(\mathbf{A}, \mathbf{C})^2 = d(\mathbf{A}, \mathbf{B})^2 + d(\mathbf{B}, \mathbf{C})^2 = 2d(\mathbf{A}, \mathbf{B})\, d(\mathbf{B}, \mathbf{C}) \cos|\angle\mathbf{ABC}|.$$

Proof. In the preceding argument, observe that in general $(\mathbf{C} - \mathbf{B}) \cdot (\mathbf{B} - \mathbf{A})$ is equal to $d(\mathbf{A}, \mathbf{B})\, d(\mathbf{B}, \mathbf{C}) \cos|\angle\mathbf{ABC}|$ by the definition of angle measurement.■

This is also a good place to include a proof for the trigonometric Law of Sines. The argument we shall give is purely algebraic, and unfortunately as such it is not well motivated. More geometrical proofs (which also relate the common ratio to other properties of a triangle) appear in the following online sites:

http://www.cut-the-knot.org/proofs/sine_cosine.shtml#law

http://mcraefamily.com/MathHelp/GeometryLawOfSinesProof.htm

[***Note:*** The proofs in these references use concepts that have not yet been introduced or are not in these notes; however, some key points appear in Exercise **III.4.4.**]

<u>**Theorem 9. (Law of Sines)**</u> *Given* $\triangle\mathbf{ABC}$, *let the lengths of its sides be given by*

$$d(\mathbf{B}, \mathbf{C}) = a, \quad d(\mathbf{C}, \mathbf{A}) = b, \quad \textit{and} \quad d(\mathbf{A}, \mathbf{B}) = c,$$

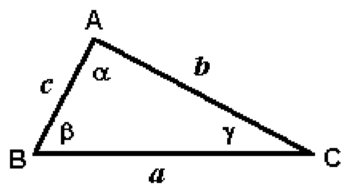
and similarly let the measures of its angles be given by given by

$$|\angle\mathbf{CAB}| = \alpha, \quad |\angle\mathbf{ABC}| = \beta, \quad \textit{and} \quad |\angle\mathbf{ACB}| = \gamma.$$

Then we have the following:

$$\frac{\text{Sin}\,\alpha}{a} = \frac{\text{Sin}\,\beta}{b} = \frac{\text{Sin}\,\gamma}{c}$$

The only property of the sine function that we shall need is that, for the values of interest to us, $\sin\theta$ is equal to the nonnegative square root of $1 - \cos^2\theta$. The notation of the theorem is completely illustrated in the diagram below.



Proof of theorem. If we can prove the first equation, then the second will follow by interchanging the roles of **A** and **C** (and hence also the roles of a and c, as well as the roles of α and γ). Note that all the lengths $a,\ b,\ c$ are positive. The first equation in the Law of Sines is equivalent to $b \sin \alpha \ = \ a \sin \beta$, and if we multiply both sides of the latter equation by c we obtain another equivalent form:

$$cb \sin \alpha \quad = \quad ca \sin \beta$$

Squaring both sides of the equation above, we see that it is equivalent to $c^2 b^2 \sin^2 \alpha \ = \ c^2 a^2 \sin^2 \beta$, and using the standard identity relating the sine and cosine functions we get the following equivalent statement:

$$c^2 b^2 (1 \ - \ \cos^2 \alpha) \quad = \quad c^2 a^2 (1 \ - \ \cos^2 \beta)$$

The latter may be written in terms of **A**, **B** and **C** as

$$||\mathbf{A} - \mathbf{B}||^2\, ||\mathbf{A} - \mathbf{C}||^2 \ - \ [(\mathbf{B} - \mathbf{A}) \cdot (\mathbf{C} - \mathbf{A})]^2 \quad =$$
$$||\mathbf{A} - \mathbf{B}||^2\, ||\mathbf{B} - \mathbf{C}||^2 \ - \ [(\mathbf{C} - \mathbf{B}) \cdot (\mathbf{A} - \mathbf{B})]^2$$

and if we make the substitutions $\mathbf{x} \ = \ \mathbf{C} - \mathbf{B}$, $\mathbf{y} \ = \ \mathbf{A} - \mathbf{C}$, $\mathbf{x} + \mathbf{y} \ = \ \mathbf{A} - \mathbf{B}$, then we can further rewrite the equation above in the following form:

$$||\mathbf{x} + \mathbf{y}||^2\, ||\mathbf{y}||^2 \ - \ [(\mathbf{x} + \mathbf{y}) \cdot \mathbf{y}]^2 \quad = \quad ||\mathbf{x} + \mathbf{y}||^2\, ||\mathbf{x}||^2 \ - \ [\mathbf{x} \cdot (\mathbf{x} + \mathbf{y})]^2$$

If we expand the left and right hand sides of this equation, we see that the preceding equation is equivalent to the following one:

$$||\mathbf{x}||^4 + 2(\mathbf{x} \cdot \mathbf{y})\, ||\mathbf{x}||^2 \ + \ ||\mathbf{x}||^2 ||\mathbf{y}||^2 \ - \ [\, ||\mathbf{x}||^2 + (\mathbf{x} \cdot \mathbf{y})\,]^2 \quad =$$
$$||\mathbf{y}||^4 + 2(\mathbf{x} \cdot \mathbf{y})\, ||\mathbf{y}||^2 \ + \ ||\mathbf{x}||^2 ||\mathbf{y}||^2 \ - \ [\, ||\mathbf{y}||^2 + (\mathbf{x} \cdot \mathbf{y})\,]^2$$

If we now simplify both sides, we find that each is equal to $||\mathbf{x}||^2\, ||\mathbf{y}||^2 \ - \ (\mathbf{x} \cdot \mathbf{y})^2$, and therefore we know that the equation above (and all the preceding ones) are true. In particular, this yields $b \sin \alpha \ = \ a \sin \beta$, which is equivalent to the Law of Sines.■

The techniques which yield the Laws of Sines and Cosines also imply the standard formulas for trigonometric functions in terms of right triangles. These formulas are generally used to define the sine and cosine functions in precalculus courses, but since our definition of these functions relies upon results from calculus, we have to verify that these standard formulas are valid.

__Theorem 9A.__ *Suppose that we are given* $\triangle$**ABC** *as above with a right angle at* **C.** *Then* $\cos |\angle \mathbf{BAC}| \ = \ b/c$ *and* $\sin |\angle \mathbf{BAC}| \ = \ a/c$.

Proof. By the Law of Cosines we have

$$a^2 = b^2 + c^2 - 2bc\cos|\angle \mathbf{BAC}|$$

and if we substitute the Pythagorean formula $c^2 = a^2 + b^2$ into the right hand side and simplify, we obtain the equation

$$0 = 2b^2 - 2bc\cos|\angle \mathbf{BAC}|.$$

Solving this for $\cos|\angle \mathbf{BAC}|$, we see that $\cos|\angle \mathbf{BAC}| = b/c$.

To derive the formula for $\sin|\angle \mathbf{BAC}|$, we note that the latter and a/c are both positive, so it suffices to show that $\sin^2|\angle \mathbf{BAC}| = a^2/c^2$. But now we have

$$\sin^2|\angle \mathbf{BAC}| = 1 - \cos^2|\angle \mathbf{BAC}| = 1 - (b^2/c^2) = (c^2 - b^2/c^2).$$

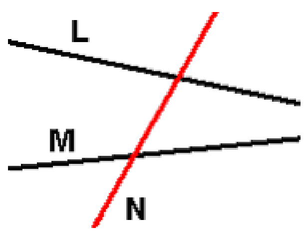
By the Pythagorean formula we know that $c^2 - b^2 = a^2$, and therefore the right hand side simplifies to a^2/c^2, and hence $\sin^2|\angle \mathbf{BAC}| = a^2/c^2$ as required.■

Transversals, parallel lines and angle sums of triangles

We shall conclude this section with a return to synthetic methods. As stated earlier, the goal is to prove the standard result about the sums of the measures of the vertex angles in a triangle.

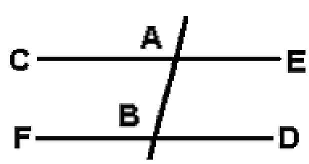
Definition. Given two coplanar lines **L** and **M**, a third line **N** in the same plane is called a ***transversal*** to **L** and **M** if it has a point in common with both of them; since the lines are supposed to be distinct, it follows that **N** has exactly one point in common with each of **L** and **M**.

The picture below describes a typical example.



In elementary geometry one has several notions of angles associated to a pair of lines cut by a transversal.

Definitions. Let **L** and **M** be distinct lines, and let **N** be a transversal meeting them in the points **B** and **A** respectively. Let **C** and **F** be points of **M** and **L** respectively which lie on the same side of **N**, and let **D** and **E** be points of **L** and **M** respectively which lie on the opposite side of **N**.



The pairs of angles $\{\angle \mathbf{CAB}, \angle \mathbf{ABD}\}$ and $\{\angle \mathbf{EAB}, \angle \mathbf{ABF}\}$ are said to be pairs of ***alternate interior angles.*** Furthermore, if we have **X**$*$**A**$*$**B** and **Y**$*$**B**$*$**A**, then the pairs of angles $\{\angle$**YBF**, $\angle$**XAE**$\}$ and $\{\angle$**XAC**, $\angle$**YBD**$\}$ are said to be pairs of ***alternate exterior angles.*** Finally, the four pairs of angles $\{\angle$**XAE**, $\angle$**ABD** $= \angle$**XBD**$\}$, $\{\angle$**XAC**, $\angle$**ABF** $= \angle$**XBF**$\}$, $\{\angle$**YBF**, $\angle$**BAC** $= \angle$**YAC**$\}$, and $\{\angle$**YBD**, $\angle$**YAE** $= \angle$**BAE**$\}$ are said to be pairs of ***corresponding angles.***

The box labeled "Parallel Lines" at the site

http://www.mathsisfun.com/geometry/alternate-interior-angles.html

gives interactive visual examples for all these types of angle pairs.

The next two results characterize Euclidean parallel lines in terms of the measures of their alternate interior angle pairs. The reasons for stating the two parts separately will become apparent in Unit **V** of these notes.

Proposition 10. *Suppose we are given the setting and notation above. If the measures of one pair of alternate interior angles are equal, then the lines* **L** *and* **M** *are parallel.*

Proof. We first claim that the measures of the other pair of alternate interior angles are also equal. For if, say, we have $|\angle$**CAB**$| = |\angle$**ABD**$|$, then the Supplement Postulate implies that $|\angle$**ABF**$| = 180° - |\angle$**ABD**$| = 180° - |\angle$**CAB**$| = |\angle$**EAB**$|$. Suppose now that the lines **L** and **M** are not parallel, and let **G** be the point where they meet. The point **G** cannot lie on the line **N,** for this would imply that **G** lies on all three lines, and we have already assumed that **L** and **M** meet **N** in different points. Suppose that **G** lies on the same side of **N** as **C** and **F.** Then we have $\angle$**ABF** $= \angle$**ABG** and also **G**$*$**A**$*$**E** (because **E** and **G** lie on opposite sides of **N**), so that $|\angle$**EAB**$| > |\angle$**ABF**$|$ by the Exterior Angle Theorem applied to $\triangle$**ABG;** but this contradicts our assumptions and observations about alternate interior angles, so it follows that **G** cannot lie on the same side of **N** as **C** and **F.** Suppose now that there is a common point **G** on the same side of **N** as **D** and **E.** Then we have $\angle$**ABD** $= \angle$**ABG** and also **G**$*$**A**$*$**C** (because **C** and **G** lie on opposite sides of **N**), so that $|\angle$**CAB**$| > |\angle$**ABD**$|$ by the Exterior Angle Theorem applied to $\triangle$**ABG;** but this contradicts our assumptions and observations about alternate interior angles, so it follows that **G** also cannot lie on the same side of **N** as **D** and **E.** Since **N** and its two sides combine to form the entire plane containing all the points and lines under consideration, it follows that there is no place in the plane that can contain a common point of **L** and **M,** and therefore these lines must be parallel.■

Proposition 11. *Suppose we are again given the setting and notation above (in particular, let* **A** *and* **B** *be the points where* **N** *meets* **M** *and* **L** *respectively), but this time assume the lines* **L** *and* **M** *are parallel. If* **C** *and* **D** *are points of* **M** *and* **L** *respectively which lie on opposite sides of* **N**, *then* $|\angle$**CAB**$| = |\angle$**ABD**$|$.

Proof. By the Protractor Postulate we know there is a unique ray **[AG** such that the corresponding open ray **(AG** lies on the same side of **N** as **C** and $|\angle$**GAB**$| = |\angle$**ABD**$|$. By the previous proposition it follows that **GA** || **L.**

By our hypotheses we also know that **M** is a line through **A** which is parallel to **L.** Since there is only one such line by Playfair's Postulate, it follows that **M** $=$ **AG.** But this means that **[AG** and **[AC** are identical and hence that $|\angle$**CAB**$| = |\angle$**ABD**$|$.■

We can summarize the preceding two results by saying that _if two lines meet a transversal in separate points, then the lines are parallel_ **if and only if** _the alternate interior angles have equal measurements_.■

Corollary 12. *Suppose in the setting above we have* **L || M.** *Then for each pair of alternate interior angles, alternate exterior angles, and corresponding angles, the two angles in the given pair have the same angular measure.*

Proof. We have already established the result for the two pairs of alternating interior angles, and we shall consider the other types of pairs according to their types.

Alternate exterior angles. Three applications of the Vertical Angle Theorem yield the following chain of equations:

$$|\angle YBF| = |\angle ABD| = |\angle CAB| = |\angle XAE|$$

Similar considerations also yield the following chain of equations:

$$|\angle XAC| = |\angle EAB| = |\angle ABF| = |\angle YBD|$$

Corresponding angles. Successive applications of the Vertical Angle Theorem, the second result on alternate interior angles, and the fact that $\angle SUV = \angle TUV$ if $T \in$ **(US**, combine to yield the following chain of equations:

$$|\angle XAE| = |\angle CAB| = |\angle ABD| = |\angle XBD|$$

Similar considerations also yield the following three chains of equations:

$$|\angle XAC| = |\angle EAB| = |\angle ABF| = |\angle YBF|$$

$$|\angle YBF| = |\angle ABD| = |\angle BAC| = |\angle XAC|$$

$$|\angle YBD| = |\angle ABF| = |\angle BAE| = |\angle YAE|$$

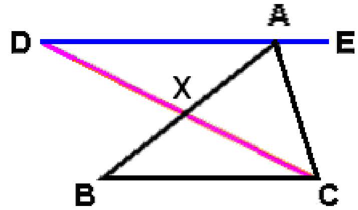
These equations cover all the pairs of alternate interior and corresponding angles listed in the definition.■

We are finally ready to state and prove the original objective of this section.

Theorem 13. *Given* $\triangle ABC$, *we have* $|\angle ABC| + |\angle BCA| + |\angle CAB| = 180°$.

Proof. We shall follow the standard argument, but we shall also verify crucial facts that are often not justified explicitly at the high school level.

Let **L** be the unique line through **A** such that **L || BC.** Then **L** contains points on both sides of **AC,** so let $D \in L$ lies on same side of **AC** as **B.** By the Crossbar Theorem we know that **(CD** meets **(AB)** at some point **X.**



Since **A∗X∗B** holds, it follows that **X** lies on the same side of **AD = L** as **C** and also lies on the same side of **BC** as **A.** Also, since **AD = L || BC,** it follows that **B** and **C** also lie on the same side of **AD = L.** Since **B, C** and **X** lie on the same side of **L = AD** we must have **C∗X∗D.** It follows that **C** and **D** must lie on opposite sides of **AB.** Likewise, **C∗X∗D** and **A∗X∗B** imply that **B, X** and **D** must lie on the same side of **AC.** Finally, since **D∗A∗E** holds, we know that **E** must lie on the opposite side of **AC** as **B, X** and **D.**

By the second proposition on alternate interior angles, we have $|\angle \mathbf{DAB}| = |\angle \mathbf{ABC}|$ and $|\angle \mathbf{EAC}| = |\angle \mathbf{ACB}|$. Now we know that **B** and **D** lie on the same side of **AC,** and since **AD = L || BC** we also know that **B** and **C** lie on the same side of **AD.** Therefore **B** lies in the interior of $\angle \mathbf{DAC}$, so that we have

$$|\angle \mathbf{DAC}| = |\angle \mathbf{DAB}| + |\angle \mathbf{BAC}| = |\angle \mathbf{ABC}| + |\angle \mathbf{BAC}|.$$

On the other hand, we also have

$$|\angle \mathbf{DAC}| = 180° - |\angle \mathbf{EAC}| = 180° - |\angle \mathbf{ACB}|.$$

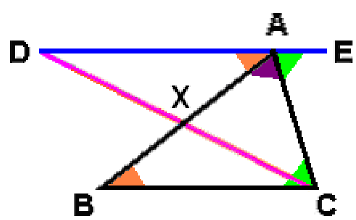
If we combine the two displayed equations we obtain

$$|\angle \mathbf{ABC}| + |\angle \mathbf{BAC}| = |\angle \mathbf{DAC}| = 180° - |\angle \mathbf{ACB}|$$

and if we rearrange terms we obtain the desired formula

$$|\angle \mathbf{ABC}| + |\angle \mathbf{BAC}| + |\angle \mathbf{ACB}| = 180°. \blacksquare$$

The following picture may be helpful for remembering the proof of Theorem **13;** in the drawing below, two angles turn out to have equal measurements if the regions near their vertices are colored with the same color.



We shall now give four standard consequences of Theorem **13:**

<u>**Corollary 14.** (Strengthened Exterior Angle Theorem)</u> *Given* △**ABC,** *let* **D** *be a point such that* **B∗C∗D.** *Then we have* $|\angle \mathbf{ACD}| = |\angle \mathbf{ABC}| + |\angle \mathbf{BAC}|$.

<u>Proof.</u> By the Supplement Postulate we have $|\angle \mathbf{BCA}| + |\angle \mathbf{ACD}| = 180°$, and hence we have $|\angle \mathbf{ABC}| + |\angle \mathbf{BCA}| + |\angle \mathbf{CAB}| = |\angle \mathbf{BCA}| + |\angle \mathbf{ACD}|$. If we subtract $|\angle \mathbf{BCA}|$ from both sides, we obtain the desired equation.■

<u>**Corollary 15.** ("Third Angles Are Equal" Theorem)</u> *Suppose we have two ordered triples of noncollinear points* **(A, B, C)** *and* **(D, E, F)** *satisfying* $|\angle \mathbf{ABC}| = |\angle \mathbf{DEF}|$ *and* $|\angle \mathbf{CAB}| = |\angle \mathbf{FDE}|$. *Then we also have* $|\angle \mathbf{ACB}| = |\angle \mathbf{DFE}|$.

Proof. By the theorem we have $|\angle \mathbf{ACB}| = 180° - |\angle \mathbf{ABC}| - |\angle \mathbf{CAB}|$ and likewise $|\angle \mathbf{DFE}| = 180° - |\angle \mathbf{DEF}| - |\angle \mathbf{FDE}|$. Since we are assuming that $|\angle \mathbf{ABC}| = |\angle \mathbf{DEF}|$ and $|\angle \mathbf{CAB}| = |\angle \mathbf{FDE}|$, it follows that we must also have $|\angle \mathbf{ACB}| = 180° - |\angle \mathbf{ABC}| - |\angle \mathbf{CAB}| = 180° - |\angle \mathbf{DEF}| - |\angle \mathbf{FDE}| = |\angle \mathbf{DFE}|$, which is what we wanted to prove.■

<u>Corollary 16. (AAS triangle congruence)</u> *Suppose we have two ordered triples of noncollinear points* **(A, B, C)** *and* **(D, E, F)** *satisfying the conditions* $d(\mathbf{B}, \mathbf{C}) = d(\mathbf{E}, \mathbf{F})$, $|\angle \mathbf{ABC}| = |\angle \mathbf{DEF}|$, *and* $|\angle \mathbf{CAB}| = |\angle \mathbf{FDE}|$. *Then* $\triangle \mathbf{ABC} \cong \triangle \mathbf{DEF}$.

Proof. By the preceding corollary we know that $|\angle \mathbf{ACB}| = |\angle \mathbf{DFE}|$. Therefore we can apply **ASA** to conclude that $\triangle \mathbf{ABC} \cong \triangle \mathbf{DEF}$.■

A much different proof of this result is mentioned in Section **V.2** (and is listed as an exercise for that section).

<u>Corollary 17.</u> *An isosceles triangle* $\triangle \mathbf{ABC}$ *is equilateral if and only if (at least) one of the angle measurements* $|\angle \mathbf{ABC}|$, $|\angle \mathbf{BCA}|$ *or* $|\angle \mathbf{CAB}|$ *is equal to* $60°$, *and in this case* **<u>ALL</u>** *of the angle measurements above are equal to* $60°$.

Proof. Since an equilateral triangle is equiangular, we know that if $\triangle \mathbf{ABC}$ is equilateral then $|\angle \mathbf{ABC}| = |\angle \mathbf{BCA}| = |\angle \mathbf{CAB}|$. If we substitute this into the equation $|\angle \mathbf{ABC}| + |\angle \mathbf{BCA}| + |\angle \mathbf{CAB}| = 180°$, we see that $3|\angle \mathbf{ABC}| = 180°$, so that $|\angle \mathbf{ABC}| = 60°$.

To prove the converse, first note that it suffices to consider the case where $d(\mathbf{A}, \mathbf{C}) = d(\mathbf{A}, \mathbf{B})$, for the remaining cases can be retrieved by interchanging the roles of the three vertices. Under the condition in the preceding sentence, there are two cases depending upon whether $|\angle \mathbf{CAB}| = 60°$ or $|\angle \mathbf{ABC}| = |\angle \mathbf{BCA}| = 60°$. In both cases we have $2|\angle \mathbf{ABC}| + |\angle \mathbf{CAB}| = 180°$. Therefore $|\angle \mathbf{CAB}| = 60°$ implies that $|\angle \mathbf{ABC}| = |\angle \mathbf{BCA}| = 60°$, and conversely $|\angle \mathbf{ABC}| = |\angle \mathbf{BCA}| = 60°$ implies $|\angle \mathbf{CAB}| = 60°$. In both cases it follows that $\triangle \mathbf{ABC}$ is equiangular, and hence $\triangle \mathbf{ABC}$ must also be equilateral.■

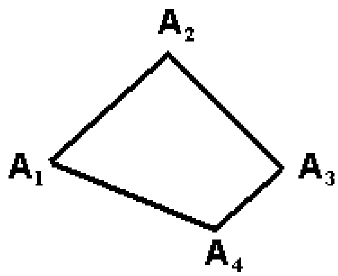
III.3 : Convex polygons

Triangles are the simplest examples of plane figures known as ***polygons.*** One way of defining the latter is to describe them as finite unions of closed segments $\mathbf{S}_k = [\mathbf{A}_k \mathbf{B}_k]$ (where $n \geq 3$ and $k = 1, \dots, n$) satisfying the following three conditions:

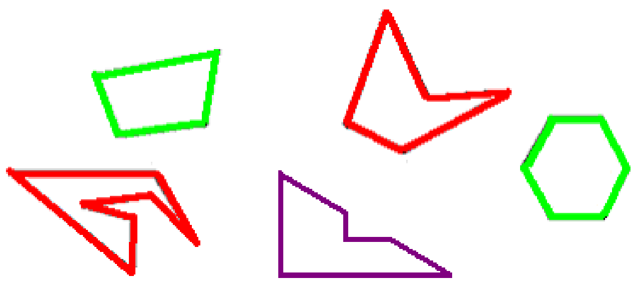
1. If $k \neq j$ then the intersection of $\mathbf{S}_j$ and $\mathbf{S}_k$ is either empty or a common endpoint.
2. If $2 \leq k \leq n$ then $\mathbf{A}_k = \mathbf{B}_{k-1}$, and also $\mathbf{B}_n = \mathbf{A}_k$.

3. For all k the sets $\{A_k, B_k = A_{k+1}, B_{k+1}\}$ and $\{A_{k-1}, B_{k-1} = A_k, B_k\}$ are noncollinear, where we take A_{n+1} to be A_1 and B_0 to be B_n.

The endpoints of the segments are called ***vertices*** of the polygon.



Five examples with $n = 4, 5, 6$ and 7 are illustrated below (for two or these examples we have $n = 6$). The labels for the vertices are omitted.



We often describe this configuration as ***polygon*** $A_1 \ldots A_n$ or $B_1 \ldots B_n$ or something similar. Frequently it is useful to define $C_k = A_k$ and B_k for arbitrary integers k by $C_k = C_s$ where $C_0 = C_n$, and more generally s is given by the long division equation $k = qn + s$, where $0 \leq s \leq n - 1$. In other words, the vertex sequences C_k are <u>*periodic*</u> and their periods are equal to n. If there are n vertices we usually say that the polygon is an ***n — gon***, and for small values of n there are often special names for these objects:

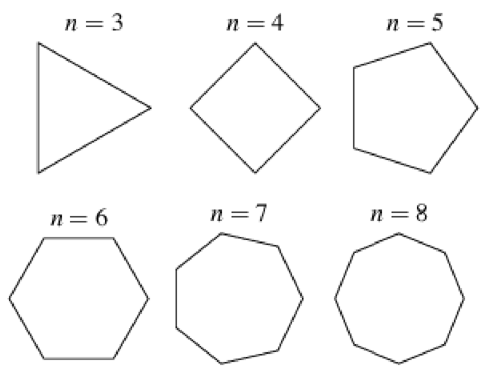
n	NAME OF POLYGON
3	***triangle***
4	***quadrilateral***
5	***pentagon***
6	***hexagon***
7	***heptagon***
8	***octagon***
9	***nonagon***
10	***decagon***
12	***dodecagon***
15	***pentadecagon***

In elementary Euclidean geometry, one special type of polygon is particularly important.

Definition. Let $A_1 \ldots A_n$ be an *n − gon.* We shall say that $A_1 \ldots A_n$ is a ***convex polygon*** if the following hold:

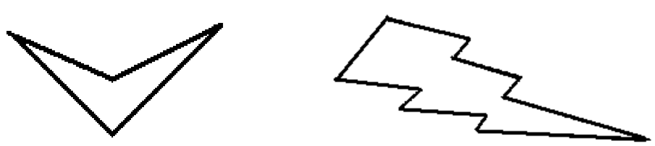
1. No three vertices are collinear.
2. For each $k = 1, \ldots, n$ all of the vertices except A_k and A_{k+1} lie on the same side of the line A_kA_{k+1} (recall our previous numbering convention that $A_{n+1} = A_1$).

In the picture above, the quadrilateral and hexagon on the right (in green) are convex, but the pentagon and heptagon (in red) and the hexagon in the middle (in purple) are not; observe that two edges of the latter are collinear, but these edges do not have any endpoints in common. Here are some additional examples, all of which are convex:



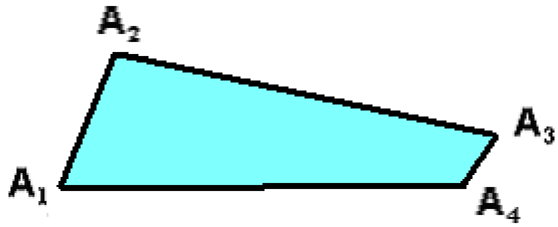
(***Source:*** **http://mathworld.wolfram.com/RegularPolygon.html**)

Note that if $n = 3$ then the second condition in the definition is vacuously true and hence *every triangle is a convex polygon*. However, for all larger values of n there are polygons that are not convex polygons; examples for $n = 4$ and $n = 11$ are depicted below.



The terminology "convex polygon" is unfortunately at odds with our earlier definition of "convex set," but unfortunately both usages are too well established to change. However, there is an important connection between the two concepts.

Definition(s). If **X, Y** and **Z** are noncollinear points and lie in the plane **P,** then $\mathbf{H}(XY, Z)$ is the half plane of all points in **P** which lie on the same side of **XY** as **Z.** Given a convex polygon $A_1 \ldots A_n$ its ***interior***, written $\mathbf{Int}\ A_1 \ldots A_n$, is the intersection of all half planes $\mathbf{H}(A_kA_{k+1}, A_{k+2})$, where A_{k+m} is defined for all integers $k + m$ by the previously stated conventions. Note that $\mathbf{H}(A_kA_{k+1}, A_{k+2}) = \mathbf{H}(A_kA_{k+1}, A_j)$ for all j such that A_j is not equal to A_k or A_{k+1}. In the picture below, the interior of $A_1A_2A_3A_4$ is the shaded region.



Since each half plane is a convex set and the intersection of convex sets is convex, it follows that $\mathrm{Int}\ \mathbf{A}_1 \dots \mathbf{A}_n$ is also ***a convex set.*** Not surprisingly, if $n = 3$ then this definition of interior reduces to the previous definition for the interior of a triangle.

Coordinate geometry and interiors of convex polygons

When the interior of a triangle was defined in Section **II.3,** a description of this region in terms of vector geometry was given in terms of barycentric coordinates. Since we now have a similar definition of interiors for convex polygons, it is natural to describe the corresponding description in terms of coordinate (or vector) geometry, and we shall now explain how to do so.

Suppose that $n \geq 3$ and $\mathbf{a}_1, \dots, \mathbf{a}_n$ (in the given order) form the vertices for a convex $n-$gon in $\mathbb{R}^2$; we shall use the previously introduced cyclic numbering convention to define $\mathbf{a}_k$ for other integral values of k, so that $\mathbf{a}_k = \mathbf{a}_{k+n}$ for all k. For each k there is a linear equation $\mathbf{u}_i \cdot \mathbf{x}_i = b_i$ which defines the line $\mathbf{a}_{i-1}\mathbf{a}_i$, and by the definition of a convex polygon we know that the $n-2$ numbers $\mathbf{u}_i \cdot \mathbf{a}_j - b_i$ (where $j = i+1, \dots, i+n-2$) all have the same sign. Replacing $\mathbf{u}_i$ and b_i with their negatives if necessary, we may assume that this sign is always positive. Therefore the interior of the convex polygon $\mathbf{A}_1 \dots \mathbf{A}_n$ is defined analytically by the finite set of strict linear inequalities $\mathbf{u}_i \cdot \mathbf{x}_i > b_i$.

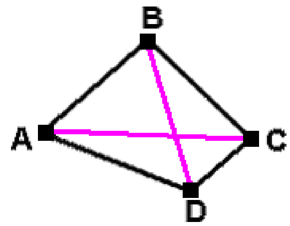
Convex quadrilaterals

Convex quadrilaterals are probably the most important class of polygons aside from triangles, and two types receive considerable attention in elementary geometry:

1. ***Parallelograms*** of the form **ABCD,** where **AB||CD** and **AD || BC;** in fact, if the parallelism conditions hold for the vertices of a polygon **ABCD** then it is automatically convex because the parallelism properties imply that the points **C** and **D** are on the same side of **AB,** the points **A** and **D** are on the same side of **BC,** the points **A** and **B** are on the same side of **CD,** and the points **B** and **C** are on the same side of **AD.**
2. ***Trapezoids*** of the form **ABCD,** where (say) **AB || CD** but **AD** is ***not*** (necessarily) parallel to **BC.** In these examples the condition for a convex quadrilateral reduces to having the points **B** and **C** on the same side of **AD,** and the points **A** and **D** on the same side of **BC** (by parallelism the other two conditions are automatically true).

The following property of convex quadrilaterals is frequently used in elementary geometry without noting the need for a logical proof:

Proposition 1. *Suppose that* **A, B, C** *and* **D** *form the vertices of a convex quadrilateral. Then the open diagonal segments* **(AC)** *and* **(BD)** *have a point in common.*



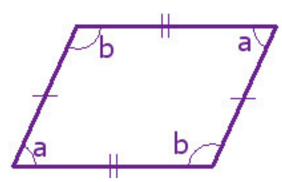
Proof. First observe that the lines **AC** and **BD** are distinct, for otherwise the four vertices would be collinear. By definition, **C** and **B** lie on the same side of **AD** and **C** and **D** lie on the same side of **AB,** so that **C** lies in the interior of **∠DAB.** Therefore the Crossbar Theorem implies that the open ray **(AC** has a point **X** in common with the open segment **(BD).**

Similarly, **A** and **D** lie on the same side of **BC** and **A** and **B** lie on the same side of **CD,** so that **A** lies in the interior of **∠BCD.** Therefore the Crossbar Theorem implies that the open ray **(CA** has a point **Y** in common with the open segment **(BD).**

Since the two lines **AC** and **BD** have at most one point in common, it follows that **X** and **Y** must be identical and this point must lie on both (**BD)** and **(AC).**■

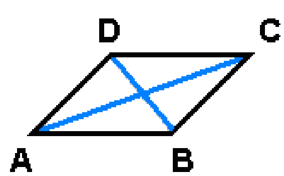
With this result at our disposal, we can derive the basic properties of parallelograms.

Proposition 2. *Suppose that* **A, B, C** *and* **D** *form the vertices of a parallelogram. Then we have* |**∠ADC**| = |**∠CDA**|, *d*(**A, B**) = *d*(**C, D**), *d*(**A, D**) = *d*(**B, C**), *and* |**∠BCD**| = |**∠DAB**|.



(***Source:*** **http://o.quizlet.com/i/ahzOyuhTFBgN--BkRypqkA_m.jpg**)

Proof. Let **X** be the point where the diagonal segments (**BD)** and **(AC)** meet. It follows that **B** and **D** lie on opposite sides of **AC,** and similarly **A** and **C** lie on opposite sides of **BD.** Therefore **{∠DCA, ∠CAB}** and **{∠DAC, ∠ACB}** are pairs of alternate interior angles.



By **ASA** we then have △**BAC** ≅ △**DCA.** In particular, this implies that |**∠ADC**| = |**∠CDA**|, *d*(**A, B**) = *d*(**C, D**), and *d*(**A, D**) = *d*(**B, C**). The other assertion of the theorem, namely $|\angle \mathbf{BCD}| = |\angle \mathbf{CAB}|$, can be proven by cyclically interchanging the roles of the vertices in the proofs; specifically, we let **B, C, D, A** take the roles of **A, B, C, D** respectively.■

Corollary 3. *In the setting of the preceding result we have*

$$|\angle \mathbf{ADC}| = |\angle \mathbf{ABC}| = 180^\circ - |\angle \mathbf{DAB}| = 180^\circ - |\angle \mathbf{DCB}|.$$

Proof. Let **E** be a point such that $\mathbf{A}*\mathbf{D}*\mathbf{E}$. Then by the results on corresponding angles and the Supplement Postulate we know that

$$|\angle \mathbf{DAB}| = |\angle \mathbf{EDC}| = 180^\circ - |\angle \mathbf{ADC}|$$

and the remaining conclusions follow from this equation and the results of the preceding theorem.■

Proposition 4. *Suppose that* **A, B, C** *and* **D** *form the vertices of a convex quadrilateral, and assume further that* **AB || CD** *and* $d(\mathbf{A}, \mathbf{B}) = d(\mathbf{C}, \mathbf{D})$. *Then the convex quadrilateral* **ABCD** *is a parallelogram.*

Proof. Once again, let **X** be the point where the diagonal segments (**BD)** and **(AC)** meet. It again follows that **B** and **D** lie on opposite sides of **AC,** and consequently $\{\angle \mathbf{DCA}, \angle \mathbf{CAB}\}$ is a pair of alternate interior angles. Since $d(\mathbf{A}, \mathbf{B}) = d(\mathbf{C}, \mathbf{D})$, by SAS we have $\triangle \mathbf{BAC} \cong \triangle \mathbf{DCA}$. Therefore we also have $|\angle \mathbf{DAC}| = |\angle \mathbf{ACB}|$. Since we already know that **B** and **D** lie on opposite sides of **AC,** it follows that we must also have **AD || BC.**■

Definition. A ***rectangle*** is a convex quadrilateral **ABCD** such that $\mathbf{AB} \perp \mathbf{BC}$, $\mathbf{BC} \perp \mathbf{CD}$, $\mathbf{CD} \perp \mathbf{AD}$ and $\mathbf{AB} \perp \mathbf{AD}$. It follows that a rectangle is automatically a parallelogram; furthermore, one can show that the fourth perpendicularity condition is redundant (this is left as an exercise to the reader). In particular, it follows immediately that ***the opposite sides of a rectangle have equal lengths.***

The following consequence of the preceding sentence is very important geometrically.

Proposition 5. *Let* **L** and **M** *be parallel lines. Let* **X** *be a point on one of these lines, let* **Y** *be a point of the other line such that* **XY** *is perpendicular to* **L** *and* **M,** *let* **Z** *be another point on one of these lines, and let* **W** *be a point of the other line such that* **ZW** *is perpendicular to* **L** *and* **M.** *Then we have* $d(\mathbf{X}, \mathbf{Y}) = d(\mathbf{Z}, \mathbf{W})$.

In everyday language, ***two parallel lines are everywhere equidistant.*** The common value of the numbers $d(\mathbf{X}, \mathbf{Y})$, $d(\mathbf{Z}, \mathbf{W})$, *etc.* is frequently called the *distance between* **L** *and* **M.**

Proof. Without loss of generality, we may as well assume that **X** lies on **L;** the proof in the case $\mathbf{X} \in \mathbf{M}$ follows by reversing the roles of **L** and **M** in the argument which follows.

Since $\mathbf{X} \in \mathbf{L}$ we also must have $\mathbf{Y} \in \mathbf{M}$. There are now a few separate cases. Let us dispose of the case where $\mathbf{Z} = \mathbf{Y}$ first. In this situation we also have $\mathbf{W} = \mathbf{X}$ and hence the distance equation is a triviality.

Suppose next that **Z** lies on **L** and is not equal to **X;** we claim that **W** is also not equal to **Y,** for if **W** = **Y** then by uniqueness of perpendiculars to a line at a point we would have that **X, Y** and **Z** would be collinear. This is impossible because the collinearity relationship would mean that the line **XZ** is perpendicular to **M,** while the hypothesis implies that **L** = **XZ** is parallel to **M.** Since two lines perpendicular to a third line are parallel, it follows that **XY || ZW,** and hence **X, Y, W** and **Z** form the vertices of a parallelogram (in that order). Therefore the basic result on parallelograms implies that d(**X, Y**) = d(**Z, W**).

Suppose now that **Z** lies on **M;** by an earlier part of the argument we know the result holds if **z** = **y,** so suppose now that they are distinct. We shall apply the reasoning of the previous paragraph systematically. First of all, if **W** is the point on **L** such that **ZW** is perpendicular to **L** and **M,** then this reasoning implies that **W** is not equal to **X.** It follows now that **XZ || YW,** and hence **X, Z, W** and **Y** form the vertices of a parallelogram (in that order). Therefore the basic result on parallelograms implies that d(**X, Y**) = d(**Z, W**).■

Of course, there are also other standard definitions of special types of parallelograms: A ***rhombus*** is a parallelogram in which the lengths of all four sides are equal, and one can define a ***square*** to be a quadrilateral that is both a rectangle and a rhombus.

We shall only mention one property of trapezoids in these notes; additional facts about them are presented in the exercises.

<u>**Proposition 6.**</u> *Suppose that* **A, B, C** *and* **D** *form the vertices of a convex quadrilateral such that* **AB || CD.** *Then* $|\angle \mathbf{DAB}| + |\angle \mathbf{ADC}| = 180°$.

<u>Proof.</u> The argument is exactly the same as the one presented in the previous corollary.■

Vector geometry and the properties of parallelograms

In the preceding discussion we have synthetic proofs for several basic theorems about parallelograms. In order to illustrate further how one uses vectors to study geometry, we shall now give some alternate proofs using vectors.

RECALL that by Exercise **I.4.3** in the file **math133exercises1.pdf**, if we are given three noncollinear points **a, b** and **d** in $\mathbb{R}^2$ and **c** = **b + d – a,** then the four points **a, b, c** and **d** (in that order) form the vertices of a parallelogram. This observation has two simple but important consequences:

$$\mathbf{b} - \mathbf{a} = \mathbf{c} - \mathbf{d} \qquad \mathbf{c} - \mathbf{b} = \mathbf{d} - \mathbf{a}$$

These follow directly from the formula for **c** given above, and they immediately imply all the basic measurement properties of a parallelogram:

<u>Vector proofs of parallelogram identities.</u> *If* **a, b, c** *and* **d** *(in that order) form the vertices of a parallelogram, then the following hold*:

1. d (**a, b**) = d (**c, d**)
2. d (**a, d**) = d (**b, c**)

3. $|\angle \mathbf{dab}| = |\angle \mathbf{bcd}|$

4. $|\angle \mathbf{dab}| + |\angle \mathbf{adc}| = 180°$

Derivations. Since $\mathbf{b} - \mathbf{a} = \mathbf{c} - \mathbf{d}$ and $\mathbf{c} - \mathbf{b} = \mathbf{d} - \mathbf{a}$, we have the following:

$$d(\mathbf{a}, \mathbf{b}) = ||\mathbf{b} - \mathbf{a}|| = ||\mathbf{c} - \mathbf{d}|| = d(\mathbf{a}, \mathbf{b})$$

$$d(\mathbf{a}, \mathbf{d}) = ||\mathbf{d} - \mathbf{a}|| = ||\mathbf{c} - \mathbf{b}|| = d(\mathbf{b}, \mathbf{c})$$

This proves the first two statements. To prove the third statement, observe that $\cos|\angle \mathbf{dab}|$ is equal to

$$\frac{(\mathbf{d} - \mathbf{a})\cdot(\mathbf{b} - \mathbf{a})}{|\mathbf{d} - \mathbf{a}|\cdot|\mathbf{b} - \mathbf{a}|} = \frac{(\mathbf{b} - \mathbf{c})\cdot(\mathbf{d} - \mathbf{c})}{|\mathbf{b} - \mathbf{c}|\cdot|\mathbf{d} - \mathbf{c}|}$$

and the latter is equal to $\cos|\angle \mathbf{bcd}|$, so that $\cos|\angle \mathbf{dab}| = \cos|\angle \mathbf{bcd}|$. Since the cosine function is a strictly decreasing function on angle measurements between 0 and 180 degrees, the equation at the end of the previous sentence implies the third statement. Finally, to prove the fourth assertion we proceed much as in the immediately preceding discussion, but the outcome is slightly different. In this case $\cos|\angle \mathbf{adc}|$ is equal to

$$\frac{(\mathbf{a} - \mathbf{d})\cdot(\mathbf{c} - \mathbf{d})}{|\mathbf{a} - \mathbf{d}|\cdot|\mathbf{c} - \mathbf{d}|} = -\frac{(\mathbf{d} - \mathbf{a})\cdot(\mathbf{b} - \mathbf{a})}{|\mathbf{d} - \mathbf{a}|\cdot|\mathbf{b} - \mathbf{a}|}$$

and the right hand side is equal to $-\cos|\angle \mathbf{dab}|$, so $\cos|\angle \mathbf{adc}| = -\cos|\angle \mathbf{dab}|$. By the sum formula for the cosine function and it strictly decreasing nature between 0 and 180 degrees, we know that $\cos\alpha = -\cos\beta$ for $0 < \alpha, \beta < \pi$ if and only if α and β are supplementary, and thus the fourth statement also follows immediately.■

Here is one more standard result on parallelograms:

If **a**, **b**, **c** *and* **d** *(in that order) form the vertices of a parallelogram, then the diagonal lines* **ac** *and* **bd** *intersect at a point which is the midpoint of both* **[ac]** *and* **[bd]** *(in words, <u>the diagonals of the parallelogram bisect each other</u>).*

To prove this it is only necessary to check that $\frac{1}{2}(\mathbf{a} + \mathbf{c}) = \frac{1}{2}(\mathbf{b} + \mathbf{d})$. This can be done by noting that the formula for **c** implies

$$\tfrac{1}{2}(\mathbf{a} + \mathbf{c}) = \tfrac{1}{2}(\mathbf{a} + \mathbf{b} + \mathbf{d} - \mathbf{a}) = \tfrac{1}{2}(\mathbf{b} + \mathbf{d}).■$$

<u>REGULAR POLYGONS.</u> Perhaps the most important class of convex polygons aside from triangles and quadrilaterals is the class of ***regular n − gons;*** for $n = 3$ or 4 these are given by equilateral triangles and squares respectively. However, before we discuss these in general it will be helpful to have some auxiliary results of independent interest.

If **Q** is a point in the plane **P** and a is a positive real number, then the ***circle*** (in the plane **P**) ***with center* Q *and radius*** a is the set of all points **X** in **P** such that d**(X, Q)** = a. The first observation is extremely basic.

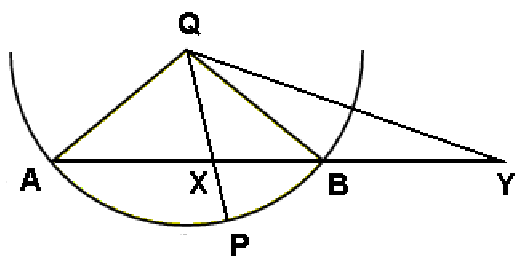
Proposition 7. *Let* **L** *be a line containing* **Q**, *let* **P** *be a plane containing* **L**, and *let* a *be a positive real number. Then there are exactly two points* **B** *and* **C** *on* **L** *that lie on the circle with center* **Q** *and radius* a, *and the center* **Q** *lies between them.*

Proof. We know there are points **X** and **Y** on **Q** such that **X*Q*Y**, and there are unique points **B** ∈ **(QX** and **C** ∈ **(QY** such that d**(B, Q)** = d**(C, Q)** = a. Since every point on **L** is either equal to **Q** or lies on one of the rays **(QX** or **(QY**, this proves that the line contains exactly two points on the circle. Furthermore, since **C** does not lie in the ray **[QX** = **[QB**, it follows that **C*Q*B** must hold.■

Here is a more substantial result.

Theorem 8. *Let* Γ *be the circle in the plane* **P** *with center* **Q** *and radius* a, and *let* **A** *and* **B** *be points on* Γ *such that* **A, B** *and* **Q** *are not collinear. Then the following are equivalent for a point* **X** ∈ **AB:**

- **(1)** **X** ∈ **(AB)** .
- **(2)** **X** ∈ **Int∠AQB.**
- **(3)** **X** *satisfies* d**(X, Q)** < a (*in everyday language*, **X** *lies* ***inside*** *the circle* Γ).



Definition. The ***interior*** of the circle Γ is the set of all points **X** in the plane of the circle such that d**(X, Q)** < a, and similarly the ***exterior*** of the circle Γ is the set of all points **X** in the plane of the circle such that d**(X, Q)** > a. Phrases like ***inside*** Γ and ***outside*** Γ are defined correspondingly, and likewise for the symbolic forms **Int Γ** and **Ext Γ.**

Proof. We shall prove that **(1)** and **(2)** are logically equivalent (each implies the other) and likewise for **(1)** and **(3).**

Verification that **(1)** *implies* **(2).** If **X** ∈ **(AB),** then **A*X*B** implies that **X** and **B** lie on the same side of **QA,** and similarly that **X** and **A** lie on the same side of **QB,** so that **X** ∈ **Int ∠AQB.**

Verification that **(2)** *implies* **(1).** If $\mathbf{X} \in \mathrm{Int}\ \angle\mathbf{AQB}$, then by the Crossbar Theorem there is a point **Y** which lies on **(AB)** and **(QX.** Since we already know that **X** lies on the line **AB,** it follows that **Y** must be **X,** and hence **A∗X∗B** is true, so that $\mathbf{X} \in$ **(AB).**

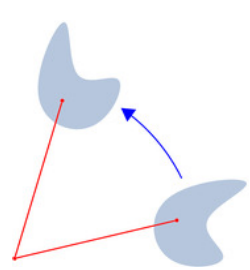
Verification that **(1)** *implies* **(3).** If $\mathbf{X} \in$ **(AB),** then **A∗X∗B** and the Exterior Angle Theorem imply that $|\angle\mathbf{AXQ}| > |\angle\mathbf{ABQ}|$; the Isosceles Triangle Theorem now implies that $|\angle\mathbf{ABQ}| = |\angle\mathbf{BAQ}| = |\angle\mathbf{XBQ}|$. Since the larger angle in $\triangle\mathbf{XQB}$ is opposite the longer side, it follows that $d(\mathbf{X}, \mathbf{Q}) < d(\mathbf{B}, \mathbf{Q}) = a$.

Verification that **(3)** *implies* **(1).** We shall prove the contrapositive. Suppose that **Y** is a point of **L** that does not lie on **(AB).** We claim that $d(\mathbf{Y}, \mathbf{Q}) \geq a$. There are four possibilities; namely, **Y** could be either **A** or **B,** we could have **A∗B∗Y,** or we could have **Y∗A∗B.** The first two cases are clear because then we have $d(\mathbf{Y}, \mathbf{Q}) = a$. For the remaining two cases, we claim it will suffice to prove the conclusion in the first case, for the other will then follow by switching the roles of **A** and **B** in the argument. We can now apply the Exterior Angle Theorem to conclude that $|\angle\mathbf{QBA}| > |\angle\mathbf{QYB}| = |\angle\mathbf{QYA}|$; the Isosceles Triangle Theorem then implies $|\angle\mathbf{QBA}| = |\angle\mathbf{QAB}| = |\angle\mathbf{QAY}|$. Since the larger angle in $\triangle\mathbf{AQY}$ is opposite the longer side, it follows that $d(\mathbf{Y}, \mathbf{Q}) < d(\mathbf{A}, \mathbf{Q}) = a$. It follows that the statements in the theorem are logically equivalent.■

Several other basic results on circles and their interior/exterior regions are presented in Section 6 of this unit.

Regular polygons and plane rotations

We shall concentrate on analyzing *standard models* for regular n − gons; any definition of an arbitrary such object should be formulated so that one can prove that an arbitrary regular n − gon will be congruent to one of the standard models. Regular polygons are very symmetric objects, and we shall use this fact to simplify and clarify the discussion at numerous points. In order to do this we shall need to work with basic isometries of $\mathbb{R}^2$ known as ***plane rotations.*** The idea of a rotation of a given angle about a given point is intuitively clear and is illustrated by the picture below.



(***Source:*** http://en.wikipedia.org/wiki/Rotation)

There is an animated model of a plane rotation at the following online site:

http://mathworld.wolfram.com/Rotation.html

One way of making the idea of a rotation mathematically precise is to use polar coordinates. Specifically, if a point is given by the polar coordinates $(r, \alpha)_{\mathbf{POLAR}}$, then counterclockwise rotation through an angle of θ should take the original point to the rotated one with coordinates given by $(r, \alpha + \theta)_{\mathbf{POLAR}}$. If we rewrite the latter using rectangular coordinates, we can obtain an explicit formula for the rectangular coordinates of the rotated point in terms of the rectangular coordinates of the old one and trigonometric functions of θ. Specifically, such a rotation is linear in the rectangular coordinates, and the matrix which represents rotation of a 2×1 column vector about the origin by a counterclockwise angle of θ is given as follows:

$$M(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

To see that this is an orthogonal matrix and hence defines an isometry of $\mathbb{R}^2$, it suffices to check that the matrix is invertible (in fact, its determinant is equal to 1) and its inverse is given by its transpose; this is easily checked and left to the reader.

We are particularly interested in rotations where $\theta = 2\pi/n$ for some integer $n > 2$. In this case the matrix $\mathbf{B} = M(2\pi/n)$ satisfies $\mathbf{B}^n = \mathbf{I}$, but no smaller positive power of $\mathbf{B}$ is equal to $\mathbf{I}$. Furthermore, in these cases we have $\mathbf{B}^k = M(2\pi k/n)$.

Let $\mathbf{e}_1$ be the usual unit vector $(1, 0)$, and let c be a positive real number. We want our standard models of regular n – gons to have the form $\mathbf{p}_1 \ldots \mathbf{p}_n$, where for every integer $k = 1, \ldots, n$ we have $\mathbf{p}_k = \mathbf{B}^{k-1}(d\mathbf{e}_1)$ for some fixed positive number d. Alternatively, in coordinates we have

$$\mathbf{p}_k = \big(d \cos(2\pi(k-1)/n),\ d \sin(2\pi(k-1)/n)\big).$$

In order to justify this definition of standard regular n – gons, we need to verify that the constructed points $\mathbf{p}_k$ are actually the vertices of a convex polygon. The use of rotations will simplify this proof substantially. In the course of the proof we shall need the following simple property of affine transformations.

<u>**Lemma 9.**</u> *Let* **T** *be an affine transformation of* $\mathbb{R}^2$, *and let* **x**, **y**, **z** *be noncollinear points in* $\mathbb{R}^2$. *Then* **T** *maps the side of* **xy** *containing* **z** *to the side of* **T(x)T(y)** *containing* **T(z).**

<u>***Proof.***</u> Using barycentric coordinates, express an arbitrary point **p** as a linear combination $a\mathbf{x} + b\mathbf{y} + c\mathbf{z}$, where $a + b + c = 1$. If **p** and **z** lie on the same side of **xy**, then c is positive. By the properties of affine transformations derived in Section II.4 we have $\mathbf{T(p)} = a\mathbf{T(x)} + b\mathbf{T(y)} + c\mathbf{T(z)}$ so that the barycentric coordinate of **T(p)** with respect to **T(z)** is also positive, and hence the two points lie on the same side of **T(x)T(y)** as required.■

<u>**Theorem 10.**</u> *If* $\mathbf{p}_1, \ldots, \mathbf{p}_n$ *are given as in the construction above, then they form the vertices of a convex polygon (when taken in the given order).*

Proof. We adopt the previous conventions about defining $\mathbf{p}_k$ for k an arbitrary integer; it follows that $\mathbf{p}_k = \mathbf{B}^{k-1}(c\,\mathbf{e}_1)$ holds for all such k.

CLAIM: By the preceding lemma and the defining identities for the points $\mathbf{p}_k$ it will suffice to prove that *the points* $\mathbf{p}_j$ *for* $j = 3, \dots, n$ *all lie on the same side of* $\mathbf{p}_1\mathbf{p}_2$. To see this, it is enough to note that the line $\mathbf{p}_k\,\mathbf{p}_{k+1}$ is the image of $\mathbf{p}_1\mathbf{p}_2$ under $\mathbf{B}^{k-1}$ and likewise the side of $\mathbf{p}_k\,\mathbf{p}_{k+1}$ containing $\mathbf{p}_{k+2}$ is the image under $\mathbf{B}^{k-1}$ of the side of $\mathbf{p}_1\mathbf{p}_2$ containing $\mathbf{p}_3$.

By construction, all the points $\mathbf{p}_k$ lie on the circle Γ centered at the origin $\mathbf{0}$ with radius equal to d, so we need to show that all of the points $\mathbf{p}_j$ for $j = 3, \dots, n$ lie on the same side of $\mathbf{p}_1\mathbf{p}_2$. Our first observation is that $\mathbf{0}$ does not lie on the line $\mathbf{p}_1\mathbf{p}_2$, for if it did then $\mathbf{0}$, $\mathbf{p}_1$ and $\mathbf{p}_2$ would be collinear, and since $\mathbf{p}_1 = d\,\mathbf{e}_1$ this would yield the false conclusion that $\mathbf{p}_2 = -d\,\mathbf{e}_1$. Thus it is meaningful to talk about the side of $\mathbf{p}_1\mathbf{p}_2$ which contains $\mathbf{0}$; we shall prove the theorem by showing that the points $\mathbf{p}_j$ for $j = 3, \dots, n$ all lie on the same side of $\mathbf{p}_1\mathbf{p}_2$ as $\mathbf{0}$. Actually, we shall prove the less direct statement that none of these points can lie on opposite side of $\mathbf{p}_1\mathbf{p}_2$ as $\mathbf{0}$.

Suppose that $\mathbf{z}$ is a point of Γ which lies on this opposite side. Then there is a point $\mathbf{x}$ which lies on $(\mathbf{0z})$ and $\mathbf{p}_1\mathbf{p}_2$. It follows that $d(\mathbf{0}, \mathbf{x}) < d(\mathbf{0}, \mathbf{z}) = d$. By the previous result on lines and circles, this means that $\mathbf{z}$ and $\mathbf{x}$ both lie in the interior of $\angle \mathbf{p}_2\mathbf{0}\mathbf{p}_1$. Therefore we also have $|\angle \mathbf{z}\mathbf{0}\mathbf{p}_1| < |\angle \mathbf{p}_2\mathbf{0}\mathbf{p}_1|$; furthermore, since $\mathbf{z}$ and $\mathbf{p}_2$ lie on the same side of $\mathbf{0}\mathbf{p}_1$, which is just the $x-$ axis, and the first coordinate of $\mathbf{p}_2$ is positive, it follows that the same holds for $\mathbf{z}$. Combining these observations, we see that $\mathbf{z}$ has the form $(d\cos\theta, d\sin\theta)$, where $0 < \theta < 2\pi/n$. None of the points $\mathbf{p}_j$ for $j = 3, \dots, n$ can be written in this manner, so it follows that they cannot lie on the opposite side of the line $\mathbf{p}_1\mathbf{p}_2$ as $\mathbf{0}$ and hence they must all lie on the same side as $\mathbf{0}$. This completes the proof that the specified points (in the given order) are the vertices of a convex polygon.■

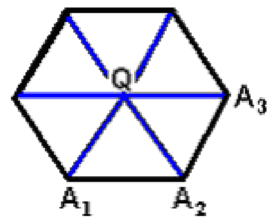
If $\mathbf{p}_1 \dots \mathbf{p}_n$ is a standard regular polygon as above, then by its rotational symmetry we know that $|\angle \mathbf{p}_1\mathbf{p}_2\mathbf{p}_3| = |\angle \mathbf{p}_k\,\mathbf{p}_{k+1}\,\mathbf{p}_{k+2}|$ for all k. We shall conclude this section by deriving the standard formula for the latter.

Proposition 11. *Given* $\mathbf{p}_1 \dots \mathbf{p}_n$ *as above, the angle measurements* $|\angle \mathbf{p}_k\,\mathbf{p}_{k+1}\mathbf{p}_{k+2}|$ *are all equal to*

$$\frac{180(n-2)}{n}.$$

For example, if $n = 5$ then the vertex angle measurements are all $108°$, if $n = 6$ then the vertex angle measurements are all $120°$ (see the drawing below), if $n = 8$ then the vertex angle measurements are all $135°$, if $n = 10$ then the vertex angle measurements are all $144°$, if $n = 60$ then the vertex angle measurements are all $174°$, and if $n = 120$ then the vertex angle measurements are all $177°$.

Proof. As noted above, by rotational symmetry it suffices to show this when $k = 1$.



To conform with the picture above, we shall denote the vertices by $\mathbf{A_1}, \ldots, \mathbf{A}_n$ and the origin by **Q**. By construction we know that $d(\mathbf{A_1}, \mathbf{Q}) = d(\mathbf{A_2}, \mathbf{Q}) = d(\mathbf{A_3}, \mathbf{Q}) = d$. Also, we have $|\angle \mathbf{A_1QA_2}| = |\angle \mathbf{A_2QA_3}| = 360°/n$. Applying the Isosceles Triangle Theorem and the result on the sum of vertex angle measurements for a triangle, we have

$$|\angle \mathbf{QA_1A_2}| = |\angle \mathbf{QA_2A_1}| = |\angle \mathbf{QA_2A_3}| = |\angle \mathbf{QA_3A_2}| = \tfrac{1}{2}(180° - (360°/n)).$$

In the course of proving that regular polygons are convex, we showed that **Q** lies on the same side of $\mathbf{A_1A_2}$ as $\mathbf{A_3}$ and also lies on the same side of $\mathbf{A_2A_3}$ as $\mathbf{A_1}$. Thus **Q** lies in the interior of $\angle \mathbf{A_1A_2A_3}$, so by the Additivity Postulate for angle measures we have

$$|\angle \mathbf{A_1A_2A_3}| = |\angle \mathbf{QA_2A_1}| + |\angle \mathbf{QA_2A_3}| = 2 \cdot [\tfrac{1}{2}(180° - (360°/n))].$$

It is a straightforward algebraic exercise to rewrite the expression on the right hand side in the form displayed in the proposition.■