Remarks on dihedral and polyhedral angles

The following pages, which are taken from an old set of geometry notes, develop the basic properties of the two basic $3-$ dimensional analogs of plane angles in a manner consistent with the setting of this course. One of the $3-$ dimensional analogs is the ***dihedral angle***, which consists of two half – planes having a common edge together with that edge. Intuitively, it looks like a piece of paper folded in the middle; this concept is discussed in Section 15.3 of Moïse. For dihedral angles, there is no vertex point as such, but instead there is an edge. There is another concept of $3-$ dimensional angle for which there is a genuine vertex point, and the simplest examples are the ***trihedral angles***. Intuitively, these look like the corners of rectangular blocks with three flat vertices joined at the common vertex or corner point, but one allows the angles of the three planar faces to take any value between 0 and 180 degrees. More generally, one can consider the corners of other solid objects as well; for example, the top of a pyramid with a square base can be viewed as defining a $4-$ faced corner, and one can do the same for the top of a pyramid whose base is an arbitrary convex polyhedron in a plane.

<u>Applications to spherical geometry.</u> If we combine Theorem 1 (the "Triangle Inequality for trihedral angles") with the standard arc length formula $s = r\theta$ for arcs in a circle of radius r, we can derive obtain one version of a fundamental result about distances between points on a sphere:

> *The shortest curve between two nonantipodal points* **A** *and* **B** *on a sphere is given by the (shorter) great circle arc joining* **A** *to* **B.**

The term "antipodal" means that the straight line joining **A** to **B** passes through the center of the sphere.

<u>Notational and bibliographic conventions</u>. One difference in notation between the following pages and the course notes needs to be mentioned; in this document the distance $d(\mathbf{A}, \mathbf{B})$ between two points **A** and **B** is denoted by $|\mathbf{AB}|$. The bibliographic references are given in the following online document:

http://math.ucr.edu/~res/math133/oldreferences.pdf

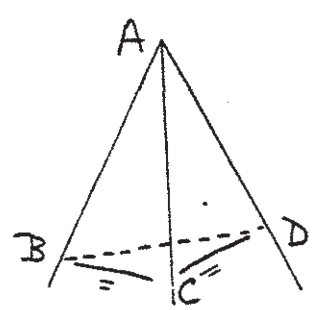
<u>Final note.</u> These pages are taken from a larger document which goes somewhat further into the subject. On the next page there is a statement about showing that there are only five types of regular polyhedra; this portion of the document has not been included here.

In this chapter we shall define trihedral and polyhedral angles, prove two fundamental inequalities on the measures of the angles determined by the plane faces,

15.1 DEFINITIONS AND FUNDAMENTAL INEQUALITIES

The most basic three-dimensional angles are dihedral angles; the reader is referred to Moise, Section 15.3 for a discussion of their basic properties, (see [Welchons and Krickenberger], Chapter II, pages 57-66, for a continuation).

In a dihedral angle, the common edge of the two half-planes can be viewed as a one-dimensional "vertex set". *Trihedral* and more generally *polyhedral* angles have zero-dimensional or point vertices. The top of a pyramid and the adjacent sides is a typical example of a polyhedral angle. One can divide polyhedral angles into two classes.



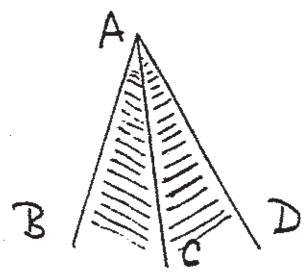
The nice ones are the *convex* angles, such as the pyramid example (a formal definition will be given later). There are also *nonconvex* polyhedral angles; roughly speaking, nonconvex polyhedral angles are to convex polyhedral angles as nonconvex polygons are to convex ones. Therefore in the formal discussion we shall only discuss convex polyhedral angles.

Just as the triangle is the simplest polygon and all triangles are convex, so also is the trihedral angle the simplex polyhedral angle, and trihedral angles are always convex. So we begin with trihedral angles.

Definition. Let A,B,C,D be four noncoplanar points.

Trihedral angle ∠A - BCD is defined to be

∠BAC ∪ ∠CAD ∪ ∠BAD ∪ Int ∠BAC ∪ Int ∠CAD ∪ Int ∠BAD.



The *faces* of the trihedral angle are the "closed interiors"

∠BAC ∪ Int ∠BAC,

∠CAD ∪ Int ∠CAD,

∠BAD ∪ Int ∠BAD.

The point A is the *vertex*, and ∠BAC, ∠CAD, ∠BAD are called the *face angles*.

Notational remark. Dihedral angles have *two* hyphens in the middle and trihedral angles have only *one*.

For reasons of space it is not possible to go through all the properties of trihedral angles that appear in the old standard solid geometry books. Many points that are intuitively clear require very complicated explanations. In any case, the following two results are both important and give information about trihedral angles that has a great deal of practical value.

THEOREM 1. *(Triangle Inequality)*. In trihedral angle $\angle A - BCD$ one has

$$|\angle BAC| + |\angle CAD| > |\angle BAD|.$$

Note. Compare this to the planar where $C \in \text{Int}\,\angle BAD$; in that case one has equality.

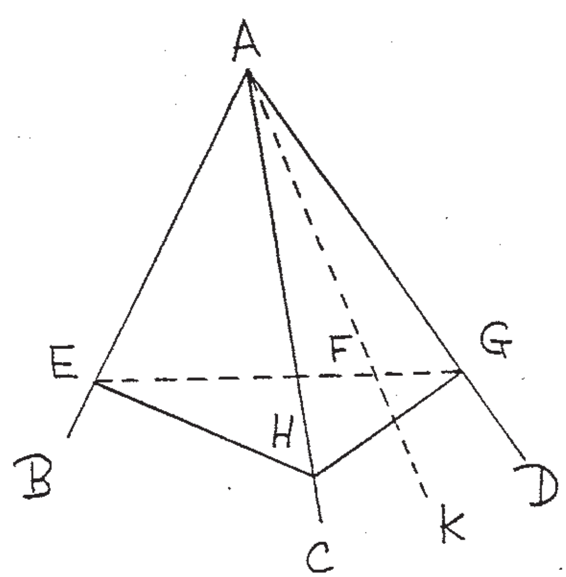
THEOREM 2. *(Angle Sum Inequality)*. In trihedral angle $\angle A - BCD$ one has

$$|\angle BAC| + |\angle CAD| + |\angle BAD| < 360°.$$

These theorems reflect a basic geometrical fact: *A set of coplanar points cannot be isometric to a set of noncoplanar points.* (Compare the discussion in Section 8.5). Physically, this means that a tripod whose legs are locked into rigid positions with respect to each other cannot be moved so that the three feet and the top all lie on a flat surface.

NOTE. *Theorem 1 and its proof are valid in neutral geometry.*

PROOF OF THEOREM 1. If $|\angle DAB| \leq |\angle CAD|$ or $|\angle DAB| \leq |\angle BAC|$ the inequality is immediate, so we may as well assume that $|\angle DAB| > |\angle CAD|$, $|\angle BAC|$.



Choose $E \in [AB$ and $G \in [AD$, and let $K \in \text{Int}\,\angle DAB$ be a point such that $|\angle KAB| = |\angle BAC|$ ($< \angle BAD|$). By the Crossbar Theorem there is a point $F \in (BD) \cap (AC$. Choose $H \in (AC$ so that $|AH| = |AF|$. Then $\Delta EAH \cong \Delta EAF$ by S.A.S., and therefore $|EH| = |EF|$.

By the Triangle Inequality (for ordinary plane triangles) and $E - F - G$ we have

$$|EH| + |HG| > |EG| = |EF| + |FG|;$$

since $|EH| = |EF|$, we conclude that $|HG| > |FG|$.

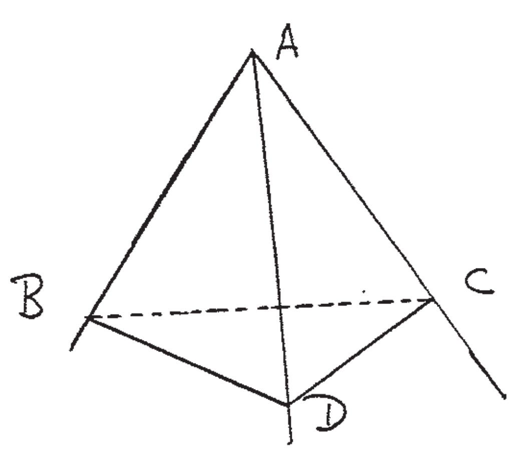
Since $|HA| = |HF|$ and $|HG| > |FG|$, the Hinge Theorem implies that $|\angle HAG| > |\angle FAG|$. On the other hand,

$$|\angle BAD| = |\angle BAF| + |\angle FAG| < |\angle BAF| + |\angle HAG|.$$

Since $|\angle BAF = \angle KAB|$ is equal to $|\angle BAC|$ and $\angle HAG = \angle CAD$, the inequality above reduces to

$$|\angle BAD| < |\angle BAC| + |\angle CAD| \blacksquare$$

PROOF OF THEOREM 2. The two main tools are Theorem 1 and the angle sum theorem for Euclidean triangles.



Consider the trihedral angles $\angle B - ACD$, $\angle C - ABD$, $\angle D - ABC$. Applying Theorem 1 to each of them, we obtain the following inequalities:

$$\text{(i)} \quad |\angle BDC| < |\angle BDA| + |\angle ADC|$$
$$\text{(ii)} \quad |\angle DCB| < |\angle DCA| + |\angle BCA|$$
$$\text{(iii)} \quad |\angle DBC| < |\angle DBA| + |\angle CBA|.$$

Since the angle-sum of a triangle is 180° we have the following equalities:

$$\text{(iv)} \quad |\angle BDC| + |\angle DCB| + |\angle DBC| = 180°$$
$$\text{(v)} \quad |\angle BAD| = 180° - |\angle ADB| - |\angle ABD|$$
$$\text{(vi)} \quad |\angle BAC| = 180° - |\angle ACB| - |\angle ABC|$$
$$\text{(vii)} \quad |\angle CAD| = 180° - |\angle ADC| - |\angle ACD|.$$

Adding (v)-(vii) together, we obtain

$$|\angle BAD| + |\angle BAC| + |\angle CAD| =$$
$$3\cdot 180 - |\angle ADB| - |\angle ABD| - |\angle ACB| - |\angle ABC| - |\angle ADC| - |\angle ACD|$$
$$= 3\cdot 180 - (|\angle ADB| - |\angle ADC|) - (|\angle BCA| + |\angle DCA|)$$
$$- (|\angle DBA| + |\angle CBA|).$$

Substitution of inequalities (i)-(iii) in the latter expression yield

$$|\angle BAD| + |\angle BAC| + |\angle CAD| < 3\cdot 180 - |\angle BDC| - |\angle DBC| - |\angle BCD|,$$

and by (iv) the right hand side is equal to $3\cdot 180 - 180 = 360$, as claimed ■

There is also a converse to these fundamental inequalities.

THEOREM 3. Let α, β, γ be three positive real numbers satisfying the following conditions:

(i) $\alpha + \beta > \gamma$, $\beta + \gamma > \alpha$, $\gamma + \alpha > \beta$.

(ii) $\alpha + \beta + \gamma < 360$.

Then there is a polyhedral angle $\angle V$-ABC such that $|\angle VBC| = \alpha$, $|\angle VCA| = \beta$, $|\angle VAB| = \gamma$ ∎

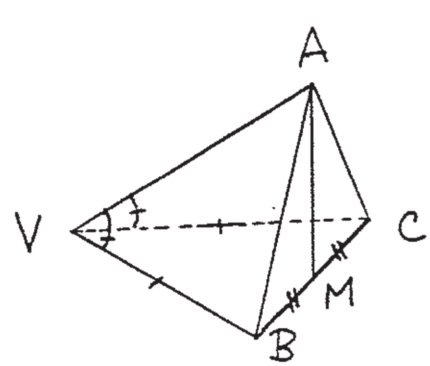
The proof will not be given here; a proof using coordinates appears in Appendix A. See [Frame] for a thorough discussion of measurement data associated to trihedral angles.

EXERCISES

A

1. The angle-sum of the face angles of a trihedral angle is 320 degrees. What is the upper limit for the measure of the largest face angle?

2. Let trihedral angle $\angle V$-ABC satisfy $|\angle AVC| = |\angle AVB|$, let $|VB| = |VC|$, and let M be the midpoint of $[BC]$. Prove that line BC is perpendicular to plane VAM.



SOLUTIONS TO ADDITIONAL EXERCISES FOR III.1 AND III.2

Here are the solutions to the exercises at the end of the file `polyangles.pdf`.

P1. Since the sum of the measures of all three face angles is at most $360°$ and the sum of two of the measures is $320°$, it follows that the measure of the third is at most $40°$.■

P2. Let Q be the plane which is the perpendicular bisector of $[BC]$, so that a point is on Q if and only if it is equidistant from B and C. It will suffice to prove that V, A, M are all equidistant from B and C; note that the three points in question cannot be collinear, for if they were then A would lie in the plane containing V, B, C.

We are given that V and M are equidistant from B and C, so we need only show that the same is true for A. Since $d(V, A) = d(V, A)$, $|\angle AVC| = |\angle AVB|$, and $d(V, B) = d(V, C)$, by **SAS** we have $\Delta AVB \cong \Delta AVC$, and this implies the desired equality $d(A, B) = d(A, C)$.■

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MORE EXERCISES ON POLYHEDRAL ANGLES

These are numerical exercises involving the fundamental inequalities for a trihedral angle.

E1. Determine whether a trihedral angle can have face angles with the following angle measures, and give reasons for your answers.

(a) 80° , 110° , 140°

(b) 72° , 128° , 156°

(c) 45° , 45° , 90°

(d) 60° , 60° , 60°

(e) 140° , 170° , 171°

(f) 105° , 118° , 130°

E2. A trihedral angle has two face angles whose measures are 80° and 120° respectively. Which of the following values can be the measure of the third face angle? Give reasons for your answer.

$$20^\circ\ ,\quad 40^\circ\ ,\quad 80^\circ\ ,\quad 90^\circ\ ,\quad 160^\circ\ ,\quad 170^\circ$$

Solutions are given on the next page.

SOLUTIONS.

E1. Each part is answered separately.

(a) Yes, because the largest angle measurement is less than the sum of the smaller two and the sum of all three angle measurements is less than $360°$.■

(b) Yes, for the same reasons as in (a).■

(c) No, because the sum of the smaller two measurements is equal to the largest measurement.■

(d) Yes, for the same reasons as in (a).■

(e) No, because the sum of all three angle measurements is greater than $360°$.■

(f) Yes, for the same reasons as in (a).■

E2. For the first three choices of the angle measure θ we have $\theta \leq 80° < 120°$, and therefore we must also have $120 < 80 + \theta$ and $200 + \theta < 360$. These imply that if $\theta \leq 80°$, then we must also have $\theta > 40°$. This means that $20°$ and $40°$ cannot be realized but $80°$ can. If $\theta = 90°$, then we have $80 \leq \theta \leq 120$ so the conditions for a trihedral angle are still $120 < 80 + \theta$ and $200 + \theta < 360$. Both of these hold if $\theta = 90$, so this value can also be realized. Finally, in the last two cases we have $80 < 120 < \theta$, and since $\theta < 180 < 120 + 80 = 200$, the Triangle Inequality condition is satisfied. However, we also have

$$\theta \; + \; 120 \; + \; 80 \quad \geq \quad 160 \; + \; 120 \; + \; 80 \quad = \quad 360$$

and therefore the second condition for realization is not met.

Summarizing, we know that the only the middle two possibilities can be realized; the first two are eliminated by the Triangle Inequality for trihedral angles, while the last two are eliminated by the constraint that the angle sum is less than $360°$.■