III.4 : Concurrence theorems

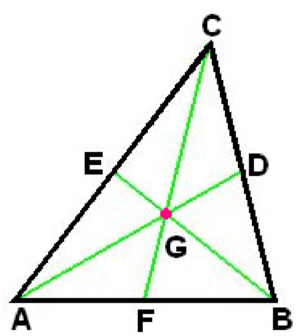
If, when three coins are tossed, they always turn up heads, we know at once that the matter demands investigation. In a like manner, if three points are always on a straight line or three lines [always] pass through a single point, we seek the reason.

The Volume Library (Educators Association, New York, 1948)

In this section ***we shall assume that all points lie in the Euclidean plane.***

If one draws three or more coplanar lines in a random manner, it is likely that no more than two will pass through a particular point. Therefore it is may seem surprising when some general method of constructing three lines always yields examples that pass through a single point. There are four basic results of this type in elementary geometry. We shall begin with one which has a simple algebraic proof.

Theorem 1. *Suppose we are given* △**ABC.** Let **D, E** and **F** *be the midpoints of the respective sides* **[BC], [AC]** *and* **[AB].** *Then the open segments* **(AD), (BE)** *and* **(CF)** *have a point in common.*



(***Source:*** http://www.algebra.com/algebra/homework/Triangles/Medians-of-a-triangle-are-concurrent.lesson)

The classical formulation of this result is that *the medians of a triangle are* ***concurrent***.

Proof. The first step is to see if the lines **AD** and **BE** have a point in common. In other words, we need to determine if there are scalars p and q such that

$$p\mathbf{D} + (1-p)\mathbf{A} = q\mathbf{E} + (1-q)\mathbf{B}$$

and if we use the midpoint formulas $\mathbf{D} = ½(\mathbf{B}+\mathbf{C})$, $\mathbf{E} = ½(\mathbf{A}+\mathbf{C})$ we obtain the following equations:

$$(1-p)\mathbf{A} + ½p\,\mathbf{B} + ½p\mathbf{C} = ½q\,\mathbf{A} + (1-q)\mathbf{B} + ½q\,\mathbf{C}$$

Equating barycentric coordinates, we conclude that $1 - p = ½q$, $1 - q = ½p$ and $½q = ½p$. The last equation implies $p = q$, and if we combine this with the others we obtain the equation $1 - p = ½p$, which implies that $p = {}^2/_3$. It is then routine to check that this value for p and q solves the equations for the barycentric coordinates and hence there is a point where **AD** and **BE** meet. In fact, since p and q lie strictly between 0 and 1, it follows that the open segments **(AD)** and **(BE)** meet, and the common point is given by

$$({}^{1}/_{3}) \cdot [\mathbf{A} + \mathbf{B} + \mathbf{C}].$$

If we apply the same argument to **(BE)** and **(CF),** we find that they also have a point in common, and the argument shows it is exactly the same point as before. Therefore it follows that this point lies on all three of the segments **(AC)**, **(BE)** and **(CF).■**

The common point is called the ***centroid*** of the triangle. By the results of Section **I.4,** this is the center of mass for a system of equal weights at each of the three vertices of the triangle (and it is also the center of mass for a triangular plate of uniform density bounded by △**ABC**). A derivation of this centroid formula is given in the online document **http://math.ucr.edu/~res/math133/triangle-centroid.pdf** .

We should also note that Theorem **1** is actually a special case of ***Ceva's Theorem*** (see Exercise **I.4.8**) with $t = u = v = ½$.

Perpendicular bisectors and altitudes

We shall need the following observation:

Lemma 2. *Let* $\mathbf{L}_1$ *and* $\mathbf{L}_2$ *be two lines that meet in one point, and let* $\mathbf{M}_1$ *and* $\mathbf{M}_2$ *be distinct lines that are perpendicular to* $\mathbf{L}_1$ *and* $\mathbf{L}_2$ *respectively. Then* $\mathbf{M}_1$ *and* $\mathbf{M}_2$ *have a point in common.*

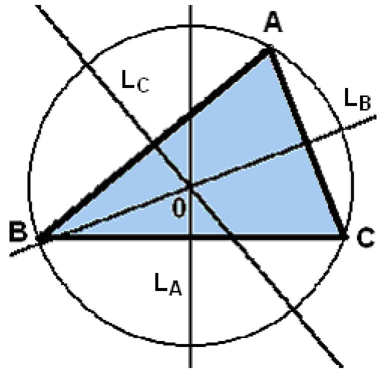
Proof. Suppose that $\mathbf{M}_1$ and $\mathbf{M}_2$ are parallel. Since $\mathbf{L}_1 \perp \mathbf{M}_1$ and $\mathbf{M}_1 \,||\, \mathbf{M}_2$ it follows that $\mathbf{L}_1 \perp \mathbf{M}_2$. However, we also have $\mathbf{L}_2 \perp \mathbf{M}_2$, so it follows that $\mathbf{L}_1 \,||\, \mathbf{L}_2$. This contradicts our assumption on $\mathbf{L}_1$ and $\mathbf{L}_2$, and therefore our assumption that $\mathbf{M}_1 \,||\, \mathbf{M}_2$ must be false, so that $\mathbf{M}_1$ and $\mathbf{M}_2$ must have a point in common.■

Theorem 3. *Given* △**ABC**, *the perpendicular bisectors of* **[BC]**, **[AC]** *and* **[AB]** *all have a point in common.*

Proof. Let $\mathbf{L}_\mathbf{A}$, $\mathbf{L}_\mathbf{B}$ and $\mathbf{L}_\mathbf{C}$ be the perpendicular bisectors of **[BC]**, **[AC]** and **[AB]** respectively. Then $\mathbf{L}_\mathbf{A} \perp \mathbf{BC}$ and $\mathbf{L}_\mathbf{B} \perp \mathbf{AC}$, and of course **AB** and **AC** have the point **C** in common. Therefore by the lemma $\mathbf{L}_\mathbf{A}$ and $\mathbf{L}_\mathbf{B}$ have a point **X** in common. Since the perpendicular bisector of a segment is the set of all points which are equidistant from the segment's endpoints, it follows that $d(\mathbf{X},\mathbf{B}) = d(\mathbf{X},\mathbf{C})$ and $d(\mathbf{X},\mathbf{A}) = d(\mathbf{X},\mathbf{C})$. Combining these, we have $d(\mathbf{X},\mathbf{A}) = d(\mathbf{X},\mathbf{C})$ and hence **X** lies on the perpendicular bisector $\mathbf{L}_\mathbf{C}$ of **[AB]**.■

The common point of the lines $\mathbf{L}_\mathbf{A}$, $\mathbf{L}_\mathbf{B}$ and $\mathbf{L}_\mathbf{C}$ is called the ***circumcenter*** of the triangle; it is the center of a (unique) circle containing **A, B** and **C.**

> Do not despair. Remember there is no triangle, however obtuse, but the circumference of some circle passes through its … vertices.
>
> S. Beckett (1906 – 1989), ***Murphy***

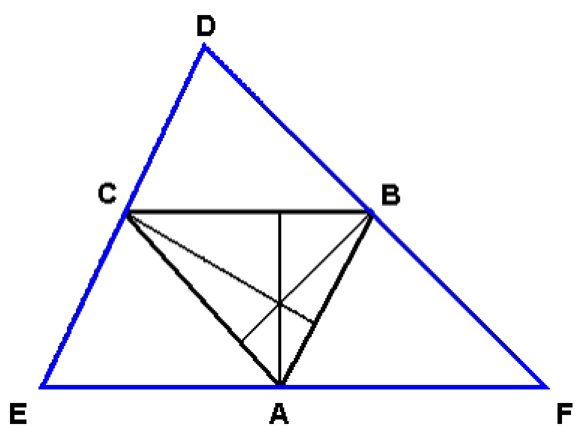


(***Source:*** **http://faculty.evansville.edu/ck6/tcenters/class/ccenter.html**)

Definition. Given $\triangle$**ABC,** the ***altitudes*** are the perpendiculars from **A** to **BC,** from **B** to **AC,** and from **C** to **AB.** Note that the points where the altitudes meet **BC, AC** and **AB** need not lie on the segments **[BC], [AC]** and **[AB].** In particular, by the results of Section 2, we know that the altitude from **A** to **BC** meets the latter in **(BC)** if and only if the vertex angles at **B** and **C** are acute (measurements less than 90 degrees).

Theorem 4. *Given* $\triangle$**ABC,** *its altitudes all have a point in common.*

Proof. The trick behind this proof is to construct a new triangle $\triangle$**DEF** such that the altitudes $\mathbf{M_A}$, $\mathbf{M_B}$ and $\mathbf{M_C}$ of $\triangle$**ABC** are the perpendicular bisectors of the sides of $\triangle$**DEF.** Since these three lines have a point in common, the result for the original triangle will follow. More precisely, one constructs the new triangle such that we have **AB** || **DE, AC** || **DF** and **BC** || **ED** and the midpoints of **[EF], [DF]** and **[DE]** are just the original vertices **A, B** and **C.** The situation is shown in the drawing below.



We must now describe the points **D, E** and **F** explicitly; we claim that they are give as follows:

$$\mathbf{D} = \mathbf{B} + \mathbf{C} - \mathbf{A} \qquad \mathbf{E} = \mathbf{A} + \mathbf{C} - \mathbf{B} \qquad \mathbf{F} = \mathbf{A} + \mathbf{B} - \mathbf{C}$$

We first need to verify that **A, B** and **C** are the midpoints of **[EF], [DF]** and **[DE]** respectively. To do this, it is only necessary to expand the three vectors $\frac{1}{2}$**(D + F)**, $\frac{1}{2}$**(E + F)**, and $\frac{1}{2}$**(D + E)** using the definitions above.

Next, we need to show that **AB || DE, AC || DF** and **BC || EF.** It will suffice to show the following:

1. The lines **AB** and **DE** are distinct, the lines **AC** and **DF** are distinct, and the lines **BC** and **EF** are distinct.

2. The difference vectors **E − D** and **B − A** are nonzero multiples of each other, the difference vectors **F − D** and **C − A** are nonzero multiples of each other, and the difference vectors **F − E** and **C − B** are nonzero multiples of each other.

We can dispose of the first item as follows: Since **C** lies on **DE** and not on **AB,** it follows that **DE** and **AB** are distinct lines; similarly, since **B** lies on **DF** and not on **AC,** it follows that **DF** and **AC** are distinct lines, and finally since **A** lies on **EF** and not on **BC,** it follows that **EF** and **BC** are distinct lines. The assertions in the second item may be checked by expanding **E − D, F − D,** and **F − E** in terms of **A, B** and **C** using the definitions. These computations yield the equations **E – D = 2(B − A), F − D = 2(C − A),** and **F − E = 2(C − B).**

Finally, we need to verify that the altitudes M_A, M_B and M_C of $\triangle$**ABC** are perpendicular to **EF, DF** and **DE** respectively. The first perpendicularity statement follows because $M_A \perp$ **BC** and **BC || EF** imply $M_A \perp$ **EF,** the second follows because $M_B \perp$ **AC** and **AC || DF** imply $M_B \perp$ **DF,** and the third follows because $M_C \perp$ **AB** and **AB || DE** imply $M_C \perp$ **DE.**■

The common point of the altitudes is called the ***orthocenter*** of the triangle.

The Euler line. The remarkable facts established above were all known to the Greek geometers. However, Euler discovered an even more amazing relationship in the 18th century; namely, ***the three concurrency points described above are always collinear.*** The line on which these points lie is called the ***Euler line*** of the triangle. Illustrations and additional information about this line appear in the following online sites:

http://faculty.evansville.edu/ck6/tcenters/class/eulerline.html

http://www.ies.co.jp/math/java/vector/veuler/veuler.html

http://en.wikipedia.org/wiki/Euler's_line

http://www.youtube.com/watch?v=CizogTmSju4&feature=related

Classical characterization of angle bisectors

We should begin by stating the basic existence and uniqueness result for angle bisectors. This was previously stated as Exercise **II.4.1,** and a proof is given in the solutions to the exercises for Section **II.4.**

Proposition 5. *Suppose that* **A, B** *and* **C** *are noncollinear points. Then there is a unique ray* **[AD** *such that* **(AD** *is contained in the interior of* $\angle$**BAC** *and* $|\angle DAB| = |\angle DAC| = \frac{1}{2}|\angle BAC|$.■

The ray **[AD** is said to ***bisect*** $\angle$**BAC** and is called the (angle) ***bisector*** of $\angle$**BAC.**

We can now state the desired characterization of angle bisectors:

Theorem 6. *Let* **A, B** *and* **C** *be noncollinear, and let* **[AD** *be the bisector of* $\angle$**BAC.** *Given a point* **X** *in* $\text{Int}\,\angle BAC$, *let* Y_X *and* Z_X *be the feet of the perpendiculars from* **X** *to* **AB** *and* **AC** *respectively. Then* **X** $\in$ **(AD** *if and only if* $d(X, Y_X) = d(X, Z_X)$.

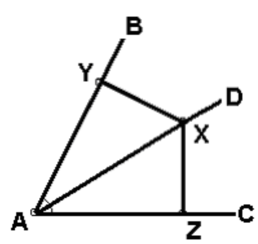
Note. If we are given a line **L** and a point **Q** not on **L,** the following standard usage we shall say that the ***foot of the perpendicular*** from **Q** to **L** is the (unique) point **S** $\in$ **L** such that $L \perp QS$.

The proof we shall give is basically standard. However, some care is needed to determine whether the points Y_X and Z_X lie on the open rays **(AB** and **(AC.** The following result will be helpful in analyzing such questions.

Lemma 7. *Let* **D** $\in$ $\text{Int}\,\angle BAC$ *and suppose that* $|\angle DAC| < 90°$. *If* **F** *is the foot of the perpendicular from* **D** *to* **AC,** *then we have* **F** $\in$ **(AC.**

Proof of Lemma. If **F** does not lie on **(AC** then either **F** = **A** or else **F*A*C** holds. But **F** = **A** implies $\angle DAC = \angle DFC$ is a right angle; since $|\angle DAC| < 90°$, this is impossible. Also, **F*A*C** implies $|\angle CAD| > |\angle FAD| = 90°$ by the Exterior Angle Theorem. Therefore we must have **F** $\in$ **(AC.**■

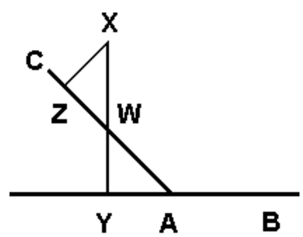
Proof of Theorem. Suppose first that **X** lies on the bisector. Since $|\angle BAC|$ is less than $180°$, it follows that both $|\angle XAB|$ and $|\angle XAC|$ are less than $90°$, so by the lemma we know that **Y** lies on **(AB** and also **Z** lies on **(AC.**



Since $|\angle XZA| = |\angle XYA| = 90°$ and $|\angle XAZ| = |\angle YAZ| = \frac{1}{2}|\angle BAC|$, we have $\triangle ZAX \cong \triangle YAX$ by **AAS,** and hence $d(X,Y) = d(X, Z)$.

Conversely, suppose that **X** $\in$ $\text{Int}\,\angle BAC$ and $d(X,Y) = d(X,Z)$. We claim that **Y** and **Z** lie on the open rays **(AB** and **(AC** respectively. Since $|\angle XAB| + |\angle XAC| = |\angle BAC| < 180°$ it follows that at least one of the terms on the left hand side must be strictly less than $90°$. Without loss of generality, we might as well assume that $|\angle XAC| < 90°$; if not, we can retrieve the result when $|\angle XAB| < 90°$ by reversing the roles of **B** and **C** and of **Y** and **Z** in the argument that follows. By the lemma the condition $|\angle XAC| < 90°$ implies that **Z** lies on **(AC.** If **Y** does not lie on **(AB,** then as in the lemma we either have $Y = A$ or else $Y{*}A{*}B$. We can dispose of the case $Y = A$ as follows: If this happens then we have a right triangle $\triangle XZA$, and since the hypotenuse is strictly longer than either of the other sides this means that $d(X,Y) = d(X,A) > d(X,Z)$, contradicting our assumption that $d(X,Y) = d(X,Z)$.

Thus it remains to eliminate the possibility that $Y{*}A{*}B$ holds. However, if $Y{*}A{*}B$ holds, then **Y** and **B** lie on opposite sides of **AC.** Since **B** and **X** lie on the same side of **AC** by hypothesis, it follows that **Y** and **X** lie on opposite sides of **AC.** Thus the line **AC** and the segment **(XY)** have some point **W** in common. It follows that $d(X, Z) > d(X,W)$. Also, since **XZ** is perpendicular to **AC** and meets the latter at **Z,** it follows (say, from the Pythagorean Theorem) that $d(X,W) \geq d(X,Z)$. Combining the observations in the preceding sentences, we have $d(X,Y) > d(X,Z)$, contradicting our assumption that these were equal. Therefore $Y{*}A{*}B$ is also impossible, and the only remaining option is for **Y** to lie on **(AB.**



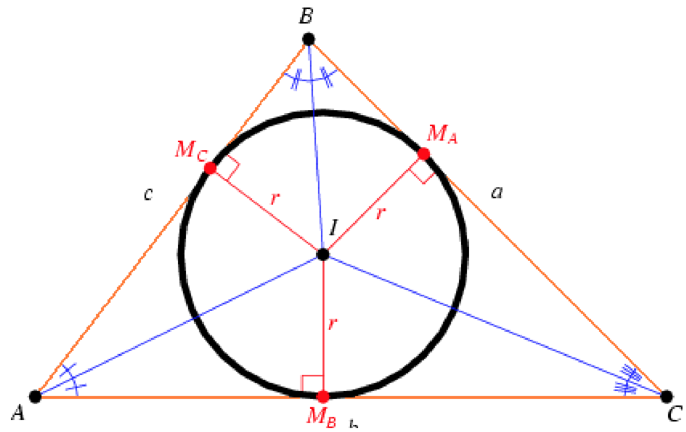
Now that we know that **Y** and **Z** lie on the open rays **(AB** and **(AC** respectively, the rest of the proof is straightforward. Triangles $\triangle XYA$ and $\triangle XZA$ are right triangles with right angles at **Y** and **Z** respectively. We know that $d(X, A) = d(X,A)$ and also $d(X,Y) = d(X,Z)$, so by the Pythagorean Theorem we also know that $d(A,Y) = d(A,Z)$. Therefore $\triangle XYA \cong \triangle XZA$ by SSS, so that $|\angle XAY| = |\angle XAZ|$. Since **Y** and **Z** lie on the open rays **(AB** and **(AC** respectively, we have $\angle XAB = \angle XAY$ and $\angle XAZ = \angle XAC$. By assumption **X** lies in the interior of $\angle BAC$, and therefore by the Additivity Postulate we have $|\angle BAC| = |\angle BAX| + |\angle XAC| = 2|\angle BAX| = 2|\angle XAC|$, so that $|\angle BAX| = |\angle XAC| = ½|\angle BAC|$, which means the ray **[AX** is the bisector of $\angle BAC$.■

The incenter

We are finally ready to state the last of the four classical concurrence theorems for triangles.

<u>Theorem 8.</u> *Given* $\triangle ABC$, *let* **[AD, [BE** and **[CF** *be the bisectors of* $\angle BAC$, $\angle ABC$ *and* $\angle BCA$ *respectively. Then the lines* **AD, BE** *and* **CF** *have a point in common, and it lies in the interior of* $\triangle ABC$.

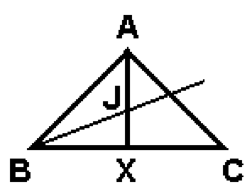
This point is called the ***incenter*** of the triangle. The reason for this name is that if one drops perpendiculars from this point to the sides of the triangle, then the feet of the perpendiculars lie on a circle inscribed within the triangle (see the illustration below).



(***Source:*** **http://mathworld.wolfram.com/Incenter.html**)

Proof. Needless to say, we are going to use the characterization of angle bisectors developed in the previous theorem.

Since **[AD** bisects **∠BAC,** the Crossbar Theorem implies there is a point **X** where **(AD** meets **(BC).** Likewise, since **[BE** bisects **∠ABC = ∠ABX,** the Crossbar Theorem also implies there is a point **J** where **(BE** meets **(AX).** Since **J** lies on **(BE,** it follows that **J** lies in the interior of **∠ABC,** and since **(AX)** is contained in **(AD,** it follows that **J** also lies in the interior of **∠BAC;** therefore **J** also lies in the interior of **△ABC.**

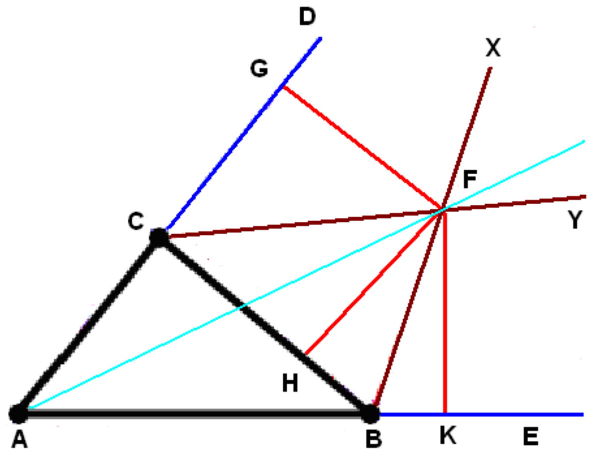


Let **T, U** and **V** be the feet of perpendiculars from **J** to **BC, AC** and **AB** respectively (these are labeled $\mathbf{M_A}$, $\mathbf{M_B}$ and $\mathbf{M_C}$ in the drawing). Since **J** lies on the bisector **[BE** we have d**(J,T) =** d**(J,V),** and since **J** lies on the bisector **(AD** we have d**(J,U) =** d**(J,V).** Combining these, we have d**(J,T) =** d**(J,U);** since we already know that **J** lies in the interior of **△ABC,** which contains the interior of **∠ACB,** it follows that **J** also lies on the bisecting ray **[CF.■**

The excenters of a triangle

There is an analog of the incenter theorem involving suitably defined ***external angle bisectors*** of a triangle.

Theorem 9. *Given a triangle* **△ABC,** *let* **D** *and* **E** *be points such that* **A∗C∗D** *and* **A∗B∗E,** *let* **[BX** *and* **[CY** *denote the bisectors of* **∠ABE** *and* **∠BCD** *respectively* *(the **external bisectors** of the angles at* **B** *and* **C**), *and let* **[AZ** *denote the bisector of* ∠**BAC.** *Then the rays* **[BX, [CY** *and* **[AZ** *are concurrent.*

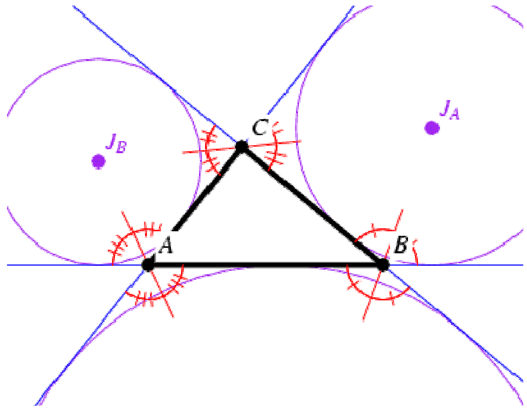


Proof. We shall first verify that **[BX** and **[CY** meet at a point **F,** and this point lies on the side of **BC** opposite **A.** The bisector conditions imply that |∠**XBC**| = ½|∠**EBC**| and |∠**YCB**| = ½|∠**GCB**|. Since both |∠**EBC**| and |∠**GCB**| are less than **180** degrees, it follows that |∠**XBC**| + |∠**YCB**| = ½|∠**EBC**| + ½|∠**GCB**| < **180°,** and therefore by the classical version of Euclid's Fifth Postulate it follows that **(BX** and **(CY** meet at a point which lies on the same side of **BC** as **Y** and **Z.** To see that this intersection point **F** lies on the side of **BC** opposite **A,** proceed as follows: The bisector condition implies that **F, X** and **E** lie on the same side of **BC,** while the betweenness condition implies that **A** and **E** lie on opposite sides of **BC,** and therefore **F** and **A** must lie on opposite sides of **BC.**

We claim that **F** lies in the interior of ∠**BAC.** Since **F** lies on **(BX** and **(CY,** it follows that **F** lies in the interiors of both ∠**ABE** and ∠**BCD.** The second of these implies that **F** and **B** lie on the same side of **CD** = **AC,** while the first implies that **F** and **C** lie on the same side of **BE** = **AB;** combining these, we obtain the assertion in the first sentence of this paragraph.

By construction the point **F** does not lie on any of the lines **AB, BC** or **AC.** Let **G, H** and **K** denote the feet of the perpendiculars from **F** to **AC, BC** and **AB** respectively. By the characterization of angle bisectors, we know that d**(F, G)** = d**(F, H)** and d**(F, H)** = d**(F, K).** Since **F** lies in the interior of ∠**BAC,** the preceding equations imply that **F** lies on the angle bisector **[AZ** of that angle. Therefore the point **F,** which by construction lies on **(BX** and **(CY,** must also lie on **(AZ,** which is what we wanted to prove.■

The point **F** is called an **excenter** for the triangle. If we interchange the roles of the vertices in the original triangle, we obtain three different excenters; the point **F** will then be the excenter corresponding to the vertex **A.** In the drawing below, the points $\mathbf{J_A}$ and $\mathbf{J_B}$ are two of the excenters, and the red lines through **A** and **B** meet in the third excenter $\mathbf{J_C}$, which is out of the picture on the side of **AB** which does not contain **C:**

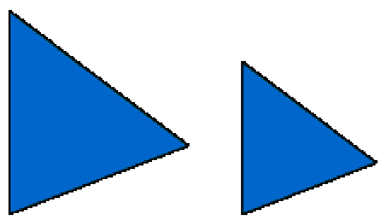


(***Source***: **http://mathworld.wolfram.com/ExteriorAngleBisector.html**)

III.5 : Similarity theorems

We shall begin by quoting a passage from **http://math.youngzones.org/similar.html**:

> Similar … [objects] are the same shape but not [necessarily] the same size. This means that corresponding angles … are congruent, and that the … [distances between corresponding points] are in the same ratio. … Similarity is found in scale models, blueprints, maps, microscopes, and when enlarging or reducing a photocopy. All of the angles are exactly the same size, so the object looks exactly like the original, only larger or smaller. … These … triangles [depicted below] have a scale factor of **3/4.**



Similarities of geometric objects are fundamental to the theory and applications of trigonometry, and similarities also have numerous applications in the other sciences and engineering.

Similarities and linear algebra

We shall be interested in the following class of mappings from $\mathbb{R}^n$ to itself:

Definition. A function (or mapping) **T** from $\mathbb{R}^n$ to itself is said to be an ***abstract similarity*** (***transformation***) if it is a $1-1$ onto map and there is a positive constant k (called the ***ratio of similitude***) such that $d(\mathbf{T}(\mathbf{x}), \mathbf{T}(\mathbf{y})) = k \cdot d(\mathbf{x}, \mathbf{y})$ for all vectors **x** and **y** in $\mathbb{R}^n$. By definition, an isometry is the same thing as an abstract similarity with ration of similitude equal to 1.

The ideas of Section II.4 yield a large family of similarities that we shall call ***regular similarities***. Specifically, if we are given a nonzero constant k and a Galilean transformation $\mathbf{G(x)} = \mathbf{Ax} + \mathbf{w}$, where $\mathbf{A}$ is orthogonal and $\mathbf{w}$ is a vector in $\mathbb{R}^n$, then the affine transformation $\mathbf{G(x)} = k\,\mathbf{Ax} + \mathbf{w}$ is an abstract similarity whose ratio of similitude is equal to $|k|$. In Section II.4 we mentioned that every isometry of $\mathbb{R}^n$ is given by a Galilean transformation, and likewise every abstract similarity of $\mathbb{R}^n$ is a regular similarity of the type described here; in fact, the result for abstract similarities is a very simple consequence of the corresponding result for isometries.

Abstract similarities and regular similarities share some basic formal properties with isometries, Galilean transformations and affine transformations. We shall merely state them; the proofs are simple modifications of the earlier arguments and are left to the reader as exercises:

Proposition 1. *The identity map is an abstract similarity from $\mathbb{R}^n$ to itself with ratio of similitude 1. If $\mathbf{T}$ is an abstract similarity from $\mathbb{R}^n$ to itself with ratio of similitude k, then its inverse $\mathbf{T}^{-1}$ is an abstract similarity of $\mathbb{R}^n$ with ratio of similitude k^{-1}. Finally, if $\mathbf{T}$ and $\mathbf{U}$ are abstract similarities from $\mathbb{R}^n$ to itself with ratios of similitude k and q respectively, then so is their composite $\mathbf{T}\circ\mathbf{U}$ is an abstract similarity of $\mathbb{R}^n$ with ratio of similitude kq.*■

Proposition 2. *The identity map is a regular similarity transformation from $\mathbb{R}^n$ to itself. If $\mathbf{T}$ is a regular similarity transformation from $\mathbb{R}^n$ to itself, then so is its inverse $\mathbf{T}^{-1}$. Finally, if $\mathbf{T}$ and $\mathbf{U}$ are regular similarities of $\mathbb{R}^n$, then so is their composite $\mathbf{T}\circ\mathbf{U}$.*■

Regular similarities also have the following important properties:

Theorem 3. *Every regular similarity transformation $\mathbf{S}$ of $\mathbb{R}^n$ has the following geometric properties:*

1. *The function $\mathbf{S}$ sends collinear points to collinear points and noncollinear points to noncollinear points.*
2. *If $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ are noncollinear points of $\mathbb{R}^n$, then $\mathbf{S}$ preserves the measurement of the angle they form; in other words, we have $|\angle\mathbf{xyz}| = |\angle\mathbf{S(x)S(y)S(z)}|$.*

Proof. The first property holds because regular similarities are affine transformations. The proof that regular similarities preserve angle measurements is similar to the proof for Galilean transformations. The cosine of $|\angle\mathbf{xyz}|$ is the quotient of $\langle\mathbf{x}-\mathbf{y},\mathbf{z}-\mathbf{y}\rangle$ by the product of the lengths $||\mathbf{x}-\mathbf{y}||\cdot||\mathbf{z}-\mathbf{y}||$. If k is the ratio of similitude for the regular similarity transformation $\mathbf{S}$, then we have

$$||\mathbf{S}(\mathbf{x})-\mathbf{S}(\mathbf{y})||\cdot||\mathbf{S}(\mathbf{z})-\mathbf{S}(\mathbf{y})|| = k^2||\mathbf{x}-\mathbf{y}||\cdot||\mathbf{z}-\mathbf{y}||$$

and therefore the proof that **S** preserves (cosines of) angles reduces to verifying that **S** and the inner product satisfy the following compatibility condition:

$$\langle \mathbf{S}(\mathbf{x}) - \mathbf{S}(\mathbf{y}), \mathbf{S}(\mathbf{z}) - \mathbf{S}(\mathbf{y}) \rangle \;=\; k^2 \langle \mathbf{x} - \mathbf{y}, \mathbf{z} - \mathbf{y} \rangle$$

By the factorization of **S** in the first sentence of the proof we have $\mathbf{S(v)} = k\mathbf{Av} + \mathbf{w}$, and it follows immediately that $\mathbf{S}(\mathbf{u}) - \mathbf{S}(\mathbf{v}) = k\mathbf{A}(\mathbf{u} - \mathbf{v})$ for all **u** and **v.** Thus we may reason as before to show that

$$\langle \mathbf{S}(\mathbf{x}) - \mathbf{S}(\mathbf{y}), \mathbf{S}(\mathbf{z}) - \mathbf{S}(\mathbf{y}) \rangle \;=\; \langle k\mathbf{A}(\mathbf{x} - \mathbf{y}), k\mathbf{A}(\mathbf{z} - \mathbf{y}) \rangle \;=$$

$$(k^2)\cdot{}^{\mathbf{T}}(\mathbf{A}(\mathbf{x} - \mathbf{y}))\,\mathbf{A}(\mathbf{z} - \mathbf{y}) \;=\; (k^2)\cdot{}^{\mathbf{T}}(\mathbf{x} - \mathbf{y})^{\mathbf{T}}\mathbf{A}\mathbf{A}(\mathbf{z} - \mathbf{y}) \;=$$

$$(k^2)\cdot{}^{\mathbf{T}}(\mathbf{x} - \mathbf{y})\,\mathbf{I}\,(\mathbf{z} - \mathbf{y}) \;=\; k^2\cdot{}^{\mathbf{T}}(\mathbf{x} - \mathbf{y})\cdot(\mathbf{z} - \mathbf{y}) \;=\; k^2 \langle \mathbf{x} - \mathbf{y}, \mathbf{z} - \mathbf{y} \rangle$$

and hence **S** must preserve angle measurements.■

Classical triangle similarities

As in the discussion of classical triangle congruences, we start with two ordered triples of noncollinear points **(A, B, C)** and **(D, E, F),** where the two sets $\{\mathbf{A, B, C}\}$ and $\{\mathbf{D, E, F}\}$ might be identical (for example, possibly $\mathbf{D} = \mathbf{B}$, $\mathbf{E} = \mathbf{C}$ and $\mathbf{F} = \mathbf{A}$). Unless otherwise noted, k denotes a positive constant.

Definition. We shall generally write $\triangle\mathbf{ABC} \sim_k \triangle\mathbf{DEF}$ and say that $\triangle\mathbf{ABC}$ and $\triangle\mathbf{DEF}$ are ***similar with ratio of similitude*** (equal to) k if the following hold:

- The corresponding lengths of the sides satisfy $d(\mathbf{D, E}) = k\cdot d(\mathbf{A, B})$, $d(\mathbf{E, F}) = k\cdot d(\mathbf{B, C})$, and $d(\mathbf{D, F}) = k\cdot d(\mathbf{A, C})$.
- The corresponding angle measurements satisfy $|\angle\mathbf{ABC}| = |\angle\mathbf{DEF}|$, $|\angle\mathbf{BAC}| = |\angle\mathbf{EDF}|$, and $|\angle\mathbf{ACB}| = |\angle\mathbf{DFE}|$.

As in the case of triangle congruence, **in the preceding definition of similarity *the orderings of the vertices are absolutely essential*.** If the precise ratio of similitude is unknown or unimportant, the subscript k is often suppressed.

Several basic properties of similarity follow immediately.

Proposition 4. *Classical triangle similarity has the following properties:*

(1) $\triangle\mathbf{ABC} \cong \triangle\mathbf{DEF}$ *if and only if* $\triangle\mathbf{ABC} \sim_1 \triangle\mathbf{DEF}$.

(2) *If* $\triangle\mathbf{ABC} \sim_k \triangle\mathbf{DEF}$, *then* $\triangle\mathbf{DEF} \sim_{1/k} \triangle\mathbf{ABC}$.

(3) *If* $\triangle\mathbf{ABC} \sim_k \triangle\mathbf{DEF}$ *and* $\triangle\mathbf{DEF} \sim_q \triangle\mathbf{TUV}$, *then* $\triangle\mathbf{ABC} \sim_{kq} \triangle\mathbf{TUV}$.■

One basic relation between classical triangle similarity and regular similarity transformations is contained in the following result:

Proposition 5. *Suppose that* **a, b, c** *are noncollinear points in* $\mathbb{R}^n$, *and let* **S** *be a regular similarity with ratio of similitude* k. *Then* **S** *maps* $\triangle$**abc** *to* $\triangle$**S(a) S(b) S(c)** *and we have* $\triangle$**abc** $\sim_k$ $\triangle$**S(a) S(b) S(c)**.

The first part follows from general properties of affine transformations, and the second preceding follows directly from the properties of regular similarities described in a previous result.■

We shall use this proposition to prove the standard similarity theorems for triangles.

Theorem 6. (SAS Similarity Theorem) *Suppose we have ordered triples* **(A, B, C)** *and* **(D, E, F)** *as above and a positive constant* k *such that* d**(D, E)** = $k \cdot d$**(A, B)**, d**(E, F)** = $k \cdot d$**(B, C)** *and* $|\angle$**ABC**$|$ = $|\angle$**DEF**$|$. *Then* $\triangle$**ABC** $\sim_k$ $\triangle$**DEF.**

Proof. Let **T** be a similarity transformation with ratio of similitude k, and consider the triangle $\triangle$**XYZ,** where **X = T(A), Y = T(B),** and **Z = T(C).** We then have $\triangle$**ABC** $\sim_k$ $\triangle$**XYZ,** and this may be combined with the hypotheses and the **SAS** Congruence Theorem to conclude that $\triangle$**XYZ** $\cong$ $\triangle$**DEF.** Therefore by the general properties of classical triangle similarity we have $\triangle$**ABC** $\sim_k$ $\triangle$**DEF.**■

Theorem 7. (AA Similarity Theorem) *Suppose we have ordered triples* **(A, B, C)** *and* **(D, E, F)** *as above satisfying* $|\angle$**ABC**$|$ = $|\angle$**DEF**$|$ *and* $|\angle$**ACB**$|$ = $|\angle$**DFE**$|$. *Then we have* $\triangle$**ABC** $\sim_k$ $\triangle$**DEF,** *where* k = d**(E, F)**$/d$**(B, C).**

Proof. Let k be defined as in the statement of the theorem, let **T** be a similarity transformation with ratio of similitude k, and consider the triangle $\triangle$**XYZ,** where **X = T(A), Y = T(B),** and **Z = T(C).** We then have $\triangle$**ABC** $\sim_k$ $\triangle$**XYZ,** and this may be combined with the hypotheses and the **ASA** Congruence Theorem to conclude that $\triangle$**XYZ** $\cong$ $\triangle$**DEF.** Therefore by the general properties of classical triangle similarity we have $\triangle$**ABC** $\sim_k$ $\triangle$**DEF.**■

Theorem 8. (SSS Similarity Theorem) *Suppose we have ordered triples* **(A, B, C)** *and* **(D, E, F)** *as above and a positive constant* k *such that* d**(D, E)** = $k \cdot d$**(A, B)**, d**(E, F)** = $k \cdot d$**(B, C)**, *and* d**(D, F)** = $k \cdot d$**(A, C)**. *Then* $\triangle$**ABC** $\sim_k$ $\triangle$**DEF.**

Proof. Let **T** be a similarity transformation with ratio of similitude k, and consider the triangle $\triangle$**XYZ,** where **X = T(A), Y = T(B),** and **Z = T(C).** We then have $\triangle$**ABC** $\sim_k$ $\triangle$**XYZ,** and this may be combined with the hypotheses and the **SSS** Congruence Theorem to conclude that $\triangle$**XYZ** $\cong$ $\triangle$**DEF.** Therefore by the general properties of classical triangle similarity we have $\triangle$**ABC** $\sim_k$ $\triangle$**DEF.**■

We can use the following to prove the following additional link between classical similarity of triangles and regular similarity transformations. It is analogous to a link between the classical notion of congruence and the general notion described in Section **II.4** of these notes.

Theorem 9. *Suppose we have* $\triangle$**ABC** *and* $\triangle$**DEF** *in the coordinate plane* $\mathbb{R}^2$ *such that* $\triangle$**ABC** $\sim_k$ $\triangle$**DEF**. *Then there is a regular similarity transformation* **S** *with ratio of similitude* k *which sends* $\triangle$**ABC** *to* $\triangle$**DEF**.

Proof. The idea is similar to the corresponding proof for congruence. We know that the pairs $\{$**B − A, C − A**$\}$ and $\{$**E − D, F − D**$\}$ form bases for $\mathbb{R}^2$. Let L be the unique linear transformation such that L(**B − A**) = **E − D** and L(**C − A**) = **F − D.** Then the argument proving the angle superposition theorem in Section **II.4** implies that $k^{-1}L$ is an orthogonal (linear) transformation; it follows that L is a similarity transformation. Now consider the similarity transformation **S** defined by

$$\mathbf{S(X)} \;=\; \mathbf{(X - A) + D} \;=\; L\mathbf{(X)} + \big(\mathbf{D} - L\mathbf{(A)}\big).$$

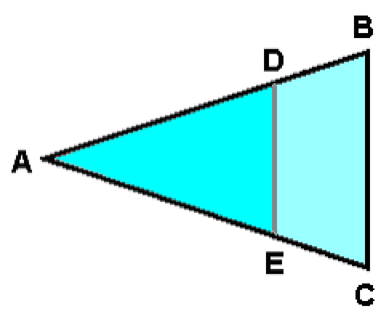
By definition and the identities **S(X) − S(Y)** = L**(X)** − L**(Y)** = L**(X − Y),** the map **S** sends **A** to **D, B** to **E** and **C** to **F,** and the ratio of similitude is equal to k. Since every similarity transformation is an affine transformation, it follows that **T** sends $\triangle$**ABC** to $\triangle$**DEF** as required.■

In analogy with congruence, the preceding result leads to **a** ***general definition of similarity for geometric figures***; namely, two geometric figures **F** and **G** are ***similar*** (in the general sense) if and only if there is a regular similarity transformation **S** which sends **F** onto **G.**

Recognizing and using similar triangles

It is often useful to have a simple criterion for recognizing similar triangles. The following one is particularly important.

Theorem 10. *Suppose we are given* $\triangle$**ABC,** *and suppose that* **D** ∈ **AB** *and* **E** ∈ **AC** *are distinct points such that* **BC || DE.** *Then* $\triangle$**ABC** $\sim$ $\triangle$**ADE**.



Proof. Write **D** = **A** + p**(B − A)** and **E** = **A** + q**(C − A)** for appropriate scalars p and q. Since **DE** is parallel to **BC,** we know that **E − D** is a nonzero multiple of **C − B,** so we shall write **E − D** = k**(C − B).** We then have the following chains of equations:

$$\mathbf{E} - \mathbf{D} \;=\; k(\mathbf{C} - \mathbf{B}) \;=\; k(\mathbf{C} - \mathbf{A}) \;-\; k(\mathbf{B} - \mathbf{A})$$

$$\mathbf{E} - \mathbf{D} \;=\; (\mathbf{E} - \mathbf{A}) \;-\; (\mathbf{D} - \mathbf{A}) \;=\; q(\mathbf{C} - \mathbf{A}) \;-\; p(\mathbf{B} - \mathbf{A})$$

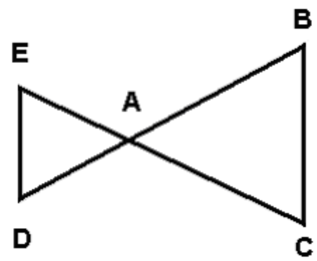
Combining these equations, we have

$$k(\mathbf{C} - \mathbf{A}) \;-\; k(\mathbf{B} - \mathbf{A}) \;=\; q(\mathbf{C} - \mathbf{A}) \;-\; p(\mathbf{B} - \mathbf{A})$$

Since **A, B** and **C** are noncollinear the vectors **B − A** and **C − A** are linearly independent, and therefore their coefficients on both sides of the equation above must be equal. Therefore we have $p = q = k$.

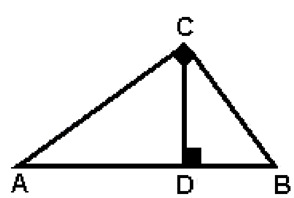
Let **T** be the regular similarity transformation given by $\mathbf{T}(\mathbf{X}) = k\mathbf{X} - k\mathbf{A} + \mathbf{A}$. By construction and previous observations we have **T(A) = A, T(B) = D,** and **T(C) = E.** Therefore it follows that $\triangle\mathbf{ABC} \sim \triangle\mathbf{ADE}$. In fact, a closer inspection of the construction implies that the ratio of similitude is equal to $|k|$.■

The appearance of the absolute value in the last sentence of the proof deserves some further comment. The drawing which appears before the proof illustrates a case where k is positive, and the drawing below depicts a case where k is negative.



The basic similarity theorems also have some standard consequences for right triangles.

<u>**Theorem 11.**</u> *Suppose that* $\triangle$**ABC** *has a right angle at* **C**, *and let* **D** *be the foot of the perpendicular from* **C** *to* **AB**. *Then* **D** *lies on the open segment* **(AB)**, *and* **AD** *splits* $\triangle$**ABC** *into two triangles, each of which is similar to* $\triangle$**ABC**. *More precisely, we have* $\triangle\mathbf{ACB} \sim \triangle\mathbf{ADC}$ *and* $\triangle\mathbf{ACB} \sim \triangle\mathbf{CDB}$.



<u>Proof.</u> The assertion that **D** lies on **(AB)** follows because both $|\angle\mathbf{CAB}|$ and $|\angle\mathbf{ABC}|$ are less than $90°$, for one can use the proof of one corollary to the Exterior Angle Theorem to conclude that these angle inequalities imply $\mathbf{D} \in (\mathbf{AB})$.

We know that $\angle\mathbf{CAD} = \angle\mathbf{BAC}$ and that $|\angle\mathbf{ACB}| = |\angle\mathbf{ADC}| = 90°$. Therefore we have $\triangle\mathbf{ACB} \sim \triangle\mathbf{ADC}$ by the **AA** similarity theorem. Likewise we know that $\angle\mathbf{DBC} = \angle\mathbf{CBA}$ and that $|\angle\mathbf{ACB}| = |\angle\mathbf{CDB}| = 90°$. Thus we also have $\triangle\mathbf{ACB} \sim \triangle\mathbf{CDB}$ by the **AA** similarity theorem.■

Corollary 12. *In the setting of the previous result we have* $d(\mathbf{C}, \mathbf{D})^2 = d(\mathbf{A},\mathbf{D})d(\mathbf{B},\mathbf{D})$.

This result is often stated in the form, "*The altitude to the hypotenuse of a right triangle is the **mean proportional** between the segments into which it divides the hypotenuse.*"

Proof. The theorem implies that $\triangle$**ADC** ~ $\triangle$**CDB,** so if k is the ratio of similitude it follows that

$$\frac{d(A,D)}{d(C,D)} = \frac{d(C,D)}{d(B,D)} = k$$

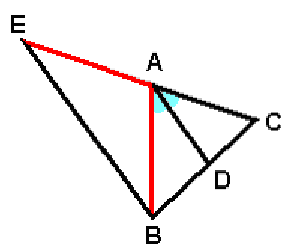
and if we clear this of fractions we obtain the equation in the corollary.■

The angle bisector theorem

We shall conclude this section with an application of similar triangles to a simple but basic question about an arbitrary triangle.

Theorem 13. (Angle Bisector Theorem) *Given* $\triangle$**ABC,** *let* **[AX** *be the bisector of* ∠**BAC,** *and let* **D** *be the point where* **(AX** *meets* **(BC)** *by the Crossbar Theorem. Then we have the following proportionality relation*:

$$\frac{d(B,A)}{d(C,A)} = \frac{d(B,D)}{d(C,D)}$$



Proof. Let **L** be the unique line through **B** which is parallel to **AD.** Since **AC** and **AD** are distinct lines, it follows that **AD** meets **L** in some point, say **E.**

We claim that **C**∗**A**∗**E** holds; this comes from the parallelism condition **BE** || **AD** and the fact that **B**∗**D**∗**C.** Since **AC** = **AE** is a transversal to the parallel lines **BE** and **AC** and **B**∗**D**∗**C** implies that **B** and **D** lie on the same side of the transversal, by the result on transversals and corresponding angles we have |∠**CAD**| = |∠**AEB**|. Furthermore, since **[BD** bisects ∠**BAC** we have |∠**BAD**| = |∠**CAD**|.

The ordering relations **C**∗**A**∗**E** and **B**∗**D**∗**C** imply that **D** and **E** lie on opposite sides of **AB.** Therefore by the result on transversals and alternate interior angles we have |∠**ABE**| = |∠**BAD**|. Combining all these, we conclude that |∠**ABE**| = |∠**AEB**|, and therefore $d(\mathbf{A},\mathbf{E}) = d(\mathbf{A},\mathbf{B})$ by the Isosceles Triangle Theorem.

The preceding observations imply that $\triangle$**CAD** ~ $\triangle$**CEB** by the **AA** Similarity Theorem. Therefore, if k is the ratio of the lengths of the sides of the second triangle to those of the first, we have the following equation:

$$\frac{d(C,E)}{d(C,A)} = \frac{d(C,B)}{d(C,D)} = k$$

If we take reciprocals of everything in the preceding display, we obtain the following:

$$\frac{d(C,A)}{d(C,E)} = \frac{d(C,D)}{d(C,B)} = \frac{1}{k}$$

Since **C*A*E** and **B*D*C** hold, we may further rewrite these as follows:

$$\frac{d(C,A)+d(A,E)}{d(C,A)} = \frac{d(C,E)}{d(C,A)} = \frac{d(C,B)}{d(C,D)} = \frac{d(C,D)+d(D,B)}{d(C,D)}$$

$$1+\frac{d(A,E)}{d(C,A)} = \frac{d(C,A)+d(A,E)}{d(C,A)} = \frac{d(C,D)+d(D,B)}{d(C,D)} = 1+\frac{d(B,D)}{d(C,D)}$$

Subtracting **1** from both sides of the outside expressions, we obtain the following:

$$\frac{d(E,A)}{d(C,A)} = \frac{d(B,D)}{d(C,D)}$$

Finally, if we combine the preceding with d(**A,E**) = d(**A,B**), we obtain

$$\frac{d(B,A)}{d(C,A)} = \frac{d(B,D)}{d(C,D)}$$

which is the equation in the statement of the Theorem.■

III.6 : Circles and constructions

In classical Euclidean geometry, there is a great deal of emphasis on constructing objects using an ***unmarked straightedge*** (**NOT a marked ruler!**) and a ***collapsible compass***. The strong preference for such constructions apparently goes back to Plato, possibly because use of other tools emphasized practicality rather than "ideas" which he regarded as more important, and the constructions in Euclid's ***Elements*** are all of this restricted type. In any case, each such classical construction involves a sequence of ***elementary steps,*** and the following two types are of particular interest in this section.

- Given a line and a circle, take the point or points where they meet.
- Given two circles, take the point or points where they meet.

Of course, such steps can be carried out only if we know one or more points where the two curves meet. In many cases, it seems clear that such common points will exist ***physically,*** but we also need to justify their existence ***mathematically.*** We have already noted that such a mathematical verification is lacking in the first proposition of Book **I** in the ***Elements***, which describes the construction of an equilateral triangle whose sides have a given length. The main purpose of this section is to develop the results on intersections of lines and circles that are needed to justify the construction steps listed above. We shall then use these results to analyze a basic construction question. Further information on constructions with straightedge and compass can be found at the following online sites:

http://en.wikipedia.org/wiki/Compass_and_straightedge

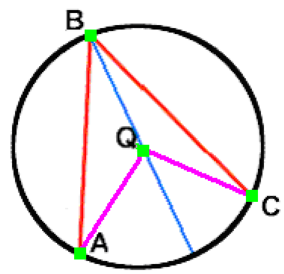
http://www.sonoma.edu/users/w/wilsonst/Courses/Math_150/c-s/default.html

http://mathworld.wolfram.com/GeometricConstruction.html

http://mathworld.wolfram.com/GeometricProblemsofAntiquity.html

http://en.wikipedia.org/wiki/Proof_of_impossibility

Angles and intercepted arcs. There are several interesting and important results on circles in elementary geometry, many of which involve one or two arcs on a circle which lie inside a given angle or pair of angles, and the relations between the measurements of these angles and the degree (or radian) measures of their intercepted arcs. The most basic example is illustrated below; in this drawing the angle $\angle ABC$ intercepts a circular arc with endpoints **A** and **C** whose measure is $|\angle AQC|$, and the latter quantity is equal to $2|\angle ABC|$.



For reasons of space, we shall not cover such material in these notes, but additional information on intercepted arcs is available in Chapter 16 of the previously cited book by Moïse and in Chapter 14 of the following classic geometry text:

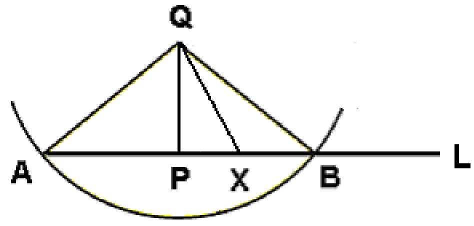
E. E. Moise and F. L. Downs, ***Geometry.*** Addison – Wesley, Reading, MA, 1964.

Throughout this section, ***unless noted otherwise all points are assumed to lie in the plane*** $\mathbb{R}^2$.

The basic theorems

We shall begin with two results on lines and circles. Neither should be surprising, but that does not eliminate the need for proofs.

Theorem 1. (Line – Circle Theorem) *Let* **L** *be a line, let* Γ *be a circle, and suppose that* **L** *contains a point inside* Γ*. Then* **L** *meets* Γ *in exactly two points.*



Proof. Let k denote the radius of Γ. It will be convenient to split the proof into two cases. Suppose first that the line **L** contains the center of Γ. Then by earlier results we know that **L** meets Γ in two points.

Suppose now that **L** does not contain the center **Q** of Γ and let **X** be a point of **L** which lies inside Γ. Let **P** be the foot of the perpendicular from **Q** to **L.** Then by (say) the Pythagorean theorem we know that d**(Q, P)** $\leq$ d**(Q, X),** which is less than k, and therefore we know that **P** also lies inside the circle. There are exactly two points **A** and **B** on **L** whose distance from **P** is equal to $\mathrm{sqrt}(k^2 - d(\mathbf{Q}, \mathbf{P})^2)$, and by the Pythagorean Theorem it follows that d**(A, Q)** $=$ d**(B, Q)** $= k$. Thus **L** meets Γ in at least two points.

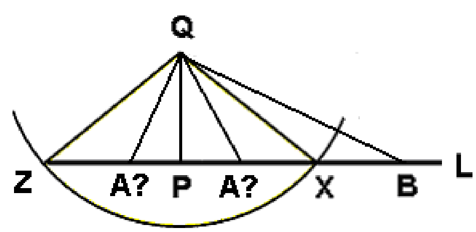
To see that these are the only points, suppose that **C** $\in$ **L** also satisfies d**(C, Q)** $= k$. Then the Pythagorean Theorem implies that

$$d(\mathbf{C}, \mathbf{P}) \quad = \quad d(\mathbf{A}, \mathbf{P}) \quad = \quad d(\mathbf{B}, \mathbf{P}) \quad = \quad \mathrm{sqrt}(k^2 - d(\mathbf{Q}, \mathbf{P})^2)$$

and since **A** and **B** are the only two points on the line **L** at this distance from **P** it follows that **C** must be either **A** or **B.**■

Proposition 2. *Let* Γ *be a circle, and suppose that we have points* **a** *and* **b** *that are (respectively) inside and outside* Γ. *Then the open segment* **(ab)** *meets* Γ *in exactly one point.*

Proof. As in the previous argument, let k be the radius of the circle. By the previous result the line **ab** meets the circle Γ in exactly two points. Let **p** $=$ **q** if **ab** contains the center of the circle, and let **p** be the foot of the perpendicular from **q** to **ab** if the line **ab** does not contain the center.



Suppose that **p** $=$ **a,** and consider the ray **[pb** $=$ **[ab.** By the proof of Proposition 1 there is a unique point **x** $\in$ **(pb** such that $d(\mathbf{x}, \mathbf{p}) = \mathrm{sqrt}(k^2 - d(\mathbf{q}, \mathbf{p})^2)$, and since **b** lies outside the circle we have

$$d(\mathbf{b}, \mathbf{p}) \quad = \quad \mathrm{sqrt}(d(\mathbf{q}, \mathbf{b})^2 - d(\mathbf{q}, \mathbf{p})^2) \quad > \quad \mathrm{sqrt}(k^2 - d(\mathbf{q}, \mathbf{p})^2) \quad = \quad d(\mathbf{x}, \mathbf{p}).$$

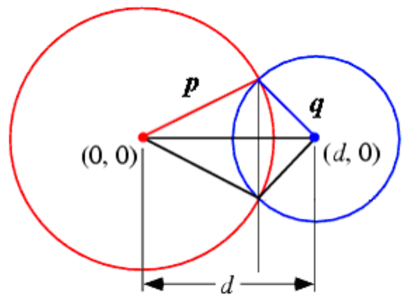
Since **x** ∈ **(pb** this means that we have the ordering **p*x*b** or equivalently **a*x*b,** so that **x** lies on **(ab).** The Pythagorean Theorem now implies that **x** lies on the original circle. Conversely, if **z** is any point on **(pb) = (ab)** which also lies on the circle, then **z** also lies on the ray **[pb** and by the Pythagorean Theorem d**(z, p)** = $\mathrm{sqrt}(k^2 - d(\mathbf{q}, \mathbf{p})^2)$ = d**(x, p),** so that **x** must be equal to **z.**

If **p** and **a** are distinct, parts of the preceding argument go through, but more work is needed. First of all, we now have d**(a, p)** < d**(x, p)** as well as d**(x, p)** < d**(b, p).** Next, there are two cases depending upon whether **[pa = [pb** or **[pa** is the opposite ray to **[pb.** Suppose first that the rays are equal. Then the distance relations imply the ordering relationship **x*a*b,** so that **x** lies on the circle and on **(ab).** Furthermore, if **z** is any such point, then **z** ∈ **(ab)** implies **z** ∈ **(pb** and the Pythagorean Theorem again implies that d**(x, q)** = d**(z, q),** so that **x = z.** Turning to the ***remaining case,*** if **a** and **b** lie on opposite rays then we have **a*p*b** as well as **p*x*b,** and these combine to show that **a*x*b,** so that **x** lies on **(ab).** Conversely, if **z** is any point on the segment and the circle, we claim that **z** lies on **[pb;** note that we have d**(z, p)** = d**(x, p)** by yet another application of the Pythagorean Theorem. If **z** does not lie on **[pb,** then we would have **z*p*b** and since **z** lies on **(ab)** we would also have **a*z*b.** Taken together, these imply **a*z*p,** so that d**(a, p)** > d**(z, p)** = d**(x, p).** But we have already proven the reverse inequality, so this is a contradiction. The problem arises from assuming that **z** is a point on the ray **[pa,** the open segment **(ab)** and the circle, and thus we see that if a point lies on **[pa** and the circle then it cannot lie on **(ab).** Therefore there is only one point which lies on both **(ab)** and the circle.■

The next theorem is similar in nature but definitely more complicated to prove.

Theorem 3. (Two Circle or Circle – Circle Theorem) *Suppose that* Γ_1 *and* Γ_2 *are two circles with different centers such that* Γ_2 *contains a point inside* Γ_1 *and a point outside* Γ_1. *Then* Γ_1 *and* Γ_2 *meet in two points, one on each side of the line joining their centers.*

Proof. Let **b** and **c** denote the centers of Γ_1 and Γ_2 respectively, let **p** and **q** be the respective radii of these circles, and let $\mathbf{u}_1$ be a vector of unit length that is a positive scalar multiple of **c − b** (specifically, multiply the latter by the reciprocal of its length d, so that $\mathbf{c} = \mathbf{b} + d\mathbf{u}_1$). Take $\mathbf{u}_2$ to be a unit vector perpendicular to $\mathbf{u}_1$. The drawing below illustrates everything when $\mathbf{u}_1$ and $\mathbf{u}_2$ are the standard unit vectors $(1, 0)$ and $(0, 1)$ respectively.



(Adapted from http://mathworld.wolfram.com/Circle-CircleIntersection.html)

Suppose we know that $\mathbf{v} = \mathbf{a} + x\mathbf{u}_1 + y\mathbf{u}_2$ lies on Γ_2; we would like to determine when $\mathbf{v}$ lies inside, on or outside the first circle Γ_1. Since

$$||\mathbf{v} - \mathbf{a}||^2 = x^2 + y^2$$

and points on the second circle satisfy

$$q^2 = ||\mathbf{v} - \mathbf{b}||^2 = (x - d)^2 + y^2$$

it follows that a point on Γ_2 lies inside, on, or outside Γ_1 depending upon whether the quantity $q^2 + 2dx - d^2$ is less than, equal to, or greater than p^2.

The minimum value of this function on the circle occurs for the smallest possible value of x, which is $d - q$, and the maximum value of this function on the circle occurs for the largest possible value of x, which is $d + q$. Since we know there are points inside and outside the circle, we know that the given minimum value must be strictly less than p^2 and the given maximum value must be strictly greater than p^2. Therefore we have the following system of equations and inequalities:

$$q^2 - 2dq + d^2 = q^2 + 2d(d - q) - d^2 < p^2 <$$
$$q^2 + 2d(d + q) - d^2 = q^2 + 2dq - d^2$$

The latter are equivalent to

$$|q - d| < p < d + q.$$

By the preceding discussion we also know that for any point which lies on both circles the coefficient x is given by $q^2 + 2dx - d^2 = p^2$, so that

$$x = (p^2 + d^2 - q^2)/2d$$

Since the coefficient y is then given by $\pm\mathbf{sqrt}(p^2 - x^2)$, we see that two solutions of the desired type will exist if and only if $|x| < p$, and since the two sides of the line joining the centers are the sets of points where y is respectively positive or negative, these two solutions will yield one point on each side of that line. The condition $|x| < p$ is equivalent to saying that

$$-2dp < p^2 + d^2 - q^2 < 2dp.$$

which in turn is equivalent to each of the next four lines:

$$-p^2 - 2dp - d^2 < -q^2 < -p^2 + 2dp - d^2$$

$$-(p+d)^2 \;<\; -q^2 \;<\; -(p-d)^2$$
$$(p-d)^2 \;<\; q^2 \;<\; (p+d)^2$$
$$|p-d| \;<\; q \;<\; p+d$$

Therefore the proof reduces to verifying the inequalities on the preceding lines.

We know that $|q-d| \;<\; p \;<\; q+d$ by earlier steps in the proof. Now $p < q+d$ implies $p-d < q$, while $q-d < p$ implies that $q < p+d$ and $d-q < p$ implies that $d-p < q$, so all the necessary inequalities are true, and this completes the proof of the theorem.■

A converse to the Classical Triangle Inequality

We shall only consider one application of the preceding theorems to construction problems.

<u>Problem.</u> *Suppose we are given three positive real numbers* a, b, *and* c *(two or more may be equal). What are the necessary and sufficient conditions for these numbers to be the lengths of the sides of a triangle?*

The Classical Triangle Inequality yields a fundamental necessary condition; namely, the sum of every pair of the numbers must be greater than the third one. Our objective is to show that any set of three numbers satisfying these simple conditions can be realized as the set of lengths of the sides of some triangle.

We can always rename a, b and c as x, y and z such that $x \geq y \geq z$, and if we do so then the conditions of the Classical Triangle Inequality simplify to the single inequality $x < y+z$ (since the other inequalities $y < x+z$ and $z < y+x$ follow immediately from the conditions $x \geq y \geq z > 0$). Thus proving the desired converse to the Classical Triangle Inequality reduces to showing the following result:

<u>Theorem 4.</u> *Suppose we are given real numbers* $x \geq y \geq z > 0$ *which satisfy the condition* $x < y+z$. *Then there is a triangle* $\triangle$**ABC** *such that* d(**B, C**) $= x$, d(**A, C**) $= y$, *and* d(**A, B**) $= z$.

<u>Proof.</u> Let **B** and **C** be arbitrary points such that d(**B, C**) $= x$, let Γ_1 be the circle with center **B** and radius y, and let Γ_2 be the circle with center **C** with radius z. We claim that the hypothesis (hence the conclusion) of the Two Circle Theorem is satisfied.

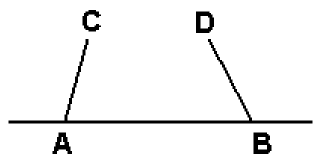
Let **U** be the point on **[BC** such that d(**B, U**) $= x+z$, and let **V** be the point on **[BC** such that d(**B, U**) $= x-z$. Then clearly $y \leq x < x+z$, and therefore the inequality $x < y+z$ implies that $x-z < y$. This means that **U** and **V** both lie on Γ_2, but **U** lies outside Γ_1 and **V** lies inside Γ_1. Therefore the Two Circle Theorem implies that Γ_1 and Γ_2 have two points in common, with one on each side of **BC.** If we take **A** to be either of these common points, then it is a routine exercise to verify that $\triangle$**ABC** satisfies the desired conditions.■

Clearly one can ask analogous questions for **SAS** and **ASA** data. The first of these is fairly easy to check (a triangle with the given measurements always exists). We shall conclude this section with the result in the **ASA** case.

The discussion depends heavily on the following partial reformulation of Euclid's original Fifth Postulate:

Theorem 5. *Let* **AB** *be a line, and let* **C** *and* **D** *be points such that* **A, B, C** *and* **D** *are coplanar and both* **C** *and* **D** *lie on the same side of* **AB.** *Then the open rays* **(AC** *and* **(BD** *have a point in common if* $|\angle CAB| + |\angle DBA| < 180°$.

As noted in Section 2 of this unit, the converse follows from the Exterior Angle Theorem.



Proof. Let **E** and **F** be points such that **C∗A∗E** and **D∗B∗F,** and let **G** be a point such that **A∗B∗G.** Then since $|\angle CAB| + |\angle DBA| < 180°$ we have

$$|\angle DBE| = 180° - |\angle DBA| > |\angle CAB|.$$

If **AC || BD,** then by the theorem on transversals and corresponding angles we would have $|\angle BDE| = |\angle CAB|$, so it follows that **AC** and **BD** must have a point in common. By construction we know that **A** and **B** are distinct, but if **AC** met **BD** on the line **AB** then **A** and **B** would have to be equal. Therefore the common point either lies on the same side of **AB** as **C** and **D,** or else the common point lies on the same side of **AB** as **E** and **F.** It suffices to eliminate the latter possibility, so suppose that **AC** meets **BD** on the same side of **AB** as **E** and **F.** Let **H** be this common point, so that **[AE = [AH** and **[BF = [BH.** By the Supplement Postulate we have

$$|\angle CAB| + |\angle BAH| = 180° = |\angle DBA| + |\angle ABH|.$$

By a corollary of the Exterior Angle Theorem we have $|\angle HAB| + |\angle HBA| < 180°$, and if we combine this with the supplementary angle equations we obtain the inequality $|\angle CAB| + |\angle DBA| > 180°$. This contradicts our initial assumption; the problem arises from the supposition that **AC** meets **BD** on the same side of **AB** as **E** and **F,** so the latter cannot happen. Therefore the lines must meet on the same side as **C** and **D.** If **S** is this half plane, then it follows that the intersection **AD ∩ BC ∩ H** is nonempty, and hence **(AC = AC ∩ S** and **(BD = BD ∩ S** must have a point in common.■

Theorem 6. *Suppose we have positive real numbers* x, α, β *such that* $\alpha + \beta < 180°$. *Then there is a triangle* $\triangle$**ABC** *with* $|\angle BCA| = \alpha$, $|\angle BAC| = \beta$, *and* $d(A, C) = x$.

Proof. Choose **A** and **C** such that $d(\mathbf{A}, \mathbf{C}) = x$. By the Protractor Postulate, there are rays **[AX** and **[CY** such that **(AX** and **(CY** lie on the same side of **AC** and their angle measurements satisfy $|\angle \mathbf{YCA}| = \alpha$ and $|\angle \mathbf{XAC}| = \beta$. Since $\alpha + \beta < 180°$, the previous result implies that **(AX** and **(CY** have a point in common. If **B** is this common point, then we have **(AX = (AB** and **(CY = (CB;** therefore $\triangle \mathbf{BAC} = \triangle \mathbf{ABC}$ will satisfy all the required conditions.■

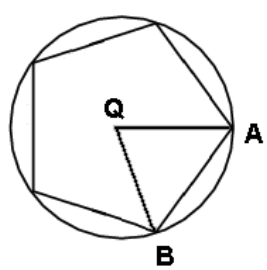
Further remarks on construction problems

For the sake of completeness, we shall mention a few other well known facts about constructions with an unmarked straightedge and (collapsible) compass. The term "collapsible" means that given two points **Q** and **P** it is possible to draw the circle with center **Q** passing through **P,** but it is not possible to lift the compass off the plane and draw a circle with some center **Q′**, ***which is* NOT *equal to* P *or* Q,** and radius equal to the distance between **P** and **Q.** Additional information on such constructions is given in Chapter 19 of the book by Moïse.

One particularly celebrated result in the ***Elements*** states that a regular polygon with 60 sides can be constructed by straightedge and compass. This requires the construction of a $6°$ angle by such means. The construction of such an angle uses three other constructions.

1. It is possible to ***bisect an angle*** by means of straightedge and compass.
2. Suppose that $0 < p, q < 180°$ and it is possible construct angles with measures p and q by straightedge and compass. **(a)** If $p + q < 180°$, then it is possible to construct an angle of measure equal to $p + q$ by means of straightedge and compass. **(b)** If $p < q$, then it is possible to construct an angle of measure $q - p$ by straightedge and compass.
3. It is possible to ***construct an equilateral triangle*** by means of straightedge and compass.
4. It is possible to ***construct a regular pentagon*** by means of straightedge and compass.

The first three of these are fairly straightforward, but the fourth requires more substantial work. From a modern viewpoint, the latter is possible for three basic reasons:



1. If, in the picture above, **Q** is the center point for the regular pentagon and **A** and **B** are adjacent vertices, then $|\angle \mathbf{AQB}| = 72°$.

2. The cosines of $72°$ and $36°$ may be written in the form $a + b\ \mathrm{sqrt}(5)$, where a and b are rational numbers (by the standard double angle formula expressing $\cos 2\theta$ as a quadratic function of $\cos \theta$, if the cosine of $36°$ is so expressible then so is the cosine of $72°$).

3. For every positive integer n and all rational numbers a and b, it is possible to construct a segment whose length is equal to $|a + b\,\mathrm{sqrt}(n)|$ (the absolute value) by means of straightedge and compass.

Detailed information on several of these points (generally at a more advanced level) is contained in the following online document:

http://math.ucr.edu/~res/math153/history02b.pdf

The latter also contains information on several other classical questions worth mentioning here, including the more general question of constructing a regular $n -$ gon by straightedge and compass and also the three classical problems from Greek geometry which turn out to be impossible to do by means of straightedge and compass (trisecting an angle, duplicating the cube, and squaring the circle).

We should also note that the concept of ***mathematical impossibility*** is often seriously misunderstood (**it is *not* the same as impossibility in engineering or technology**), and there is a discussion of this issue in the online document cited above (see also the discussion at the end of Section **II.4** in these notes). Further information on this topic may be found in the file http://math.ucr.edu/~res/math133/mathproofs.pdf or Chapter **T** of the following book:

W. Dunham, ***The Mathematical Universe: An Alphabetical Journey Through the Great Proofs, Problems, and Personalities***. Wiley, New York, 1997. ISBN: 0–471–17661–3.

Marked straightedge and compass constructions

The restriction to ***unmarked*** straghtedges in classical constructions is extremely important, and often misunderstandings arise from confusion over marked and unmarked straightedges. Although angle trisection and cube duplication cannot be done using classical (unmarked straightedge and compass) construction principles, but ancient Greek geometers did discover methods for completing these constructions with a marked straightedge and compass. Near the end of the 16th century F. Viète (or Vieta, 1540 – 1603; among other things, known for crucial advances in the development of modern symbolic mathematical notation) suggested a modified theory of constructions with the following admissible steps called ***neusis*** constructions in addition to the classical ones:

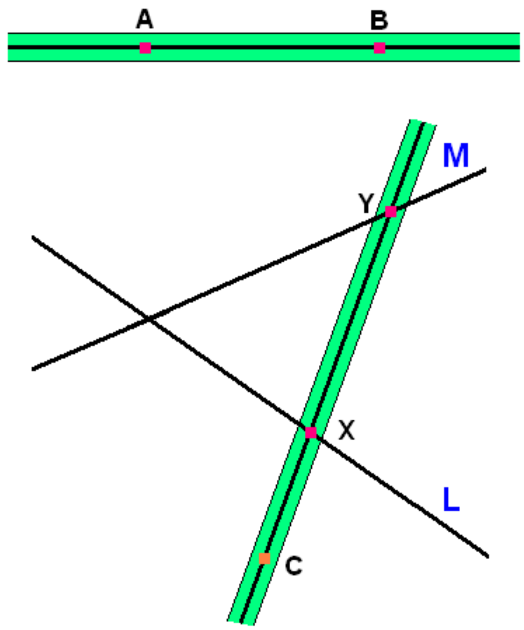
Suppose that we are given the points **A, B, C** *and the intersecting lines* **L** *and* **M** *which all lie on the same plane, and let* w *denote the distance between* **A** *and* **B.** *Then one can also construct points* **X** *and* **Y** *such that*

(1) $X \in L$ *and* $Y \in M$,

(2) *the points* **C, X,** *and* **Y** *are collinear,*

(3) *the distance between* **X** *and* **Y** *is also equal to* w.

Physically speaking, we can do this by placing the straightedge so that it passes through **A** and **B,** marking the straightedge at both points, then moving the straightedge with two marks so that it passes through **C** and one marked point lies on **L** while the other lies on **M,** and finally marking the points **X** and **Y** where the straightedge meets these two lines. Here is a drawing to illustrate such construction procedures:



There are a couple of reasons for assuming that the lines **L** and **M** are intersecting and not parallel. If **L** || **M** and **C** is an arbitrary point in the same plane as these lines, then it is not possible to find **X** and **Y** if w is less than the distance between **L** and **M,** but if w is greater than this distance then it is always possible to make the construction using classical unmarked straightedge and compass methods. On the other hand, if **L** and **M** intersect this is not necessarily the case.

The drawing is meant to suggest the following physical model for the added type of construction step: One first places the straightedge on the line **AB,** marking it at **A** and **B,** and then one moves this marked straightedge so that the line it determines passes through **C** and the two marked points lie on **L** and **M.**

To avoid further digression we shall not explain how one can use neusis constructions to trisect angles and duplicate cubes, but here are a few online references where these and other constructions are completed (for example, the construction of a regular heptagon).

http://math.berkeley.edu/~robin/Viete/construction.html

http://orion.math.iastate.edu/msm/EekhoffMSMSS07.pdf

http://www.geom.uiuc.edu/docs/forum/angtri/

http://www.uwgb.edu/dutchs/pseudosc/trisect.htm

http://www.uwgb.edu/dutchs/pseudosc/DuplCube.HTM

http://www.cut-the-knot.org/htdocs/dcforum/DCForumID4/756.shtml

Appendix A — Further topics in Euclidean geometry

> There are more things in heaven and earth,
> Horatio, than are dreamt of in your philosophy.
>
> Shakespeare, ***Hamlet,*** Act **1**, Sc. **V**, 166 – 167.

Euclid's ***Elements*** presented an integrated account of the main body of mathematical knowledge at the time, but Greek geometers had already pushed some parts of the subject considerably beyond the material covered there. Given the ***Elements'*** impact on mathematics — and indeed for civilization in general — it is not at all surprising that there has been an enormous amount of further work on its topics over the past **2300** years. In particular, during the "modern" era of mathematics beginning late in the 16^{th} century, many professional and amateur mathematicians have discovered remarkable facts about familiar figures like circles and triangles that are in the spirit of classical Greek geometry but were apparently unknown in ancient times (since many classical Greek mathematical writings have not survived and substantial parts of classical Greek mathematical work were probably never put into written form, at least some results might have been known). In Section **4** we mentioned one example; namely, the discovery of the ***Euler line.*** A detailed discussion of such results is beyond the scope of these notes, but we list some books and online references (including videos) that cover these topics.

N. Altshiller – Court, ***Modern Pure Solid Geometry.*** (2^{nd} Ed.). Chelsea Pub., New York, 1979. ISBN: 0–828–40147–0.

N. Altshiller – Court, ***College geometry: An introduction to the modern geometry of the triangle and the circle.*** (2^{nd} Enlarged Ed.). Dover, New York, 2007. ISBN: 0–486–45805–9.

A. S. Posamentier and J. Stepelman, ***Teaching Secondary School Mathematics: Techniques and Enrichment Units.*** (6^{th} Ed.). Prentice Hall, Upper Saddle River NJ, 2001. ISBN: 0–130–94514–5.

H. Perfect, ***Topics in Geometry*** (Commonwealth and International Library No. 142 ; Maths. Div. Vol. 7). Pergamon/Macmillan, London and New York, 1963.

R. D. Millman and G. D. Parker, ***Geometry — A Metric Approach with Models.*** Springer – Verlag, New York, 1990. ISBN: 0–387–97412–1.

http://jwilson.coe.uga.edu/emt669/Student.Folders/McFarland.Derelle/Euler/euler.html

http://www.cut-the-knot.org/triangle/Morley/Morley.shtml

http://www.youtube.com/watch?v=CizogTmSju4&feature=related

http://www.youtube.com/watch?v=LqF4hiNkvQk&feature=related

http://www.youtube.com/watch?v=cRebl8I0IKk&feature=channel

Section **19.4** in Moïse (*The Problem of Apollonius*) also discusses some more advanced topics in Euclidean geometry. **NOTE:** Apollonius of Perga (*c.* 262 – *c.* 190 B.C.E.) was one of the most important figures in ancient Greek mathematics, and he is especially known for his extensive study of conic sections; his contributions are described in the following online document:

http://math.ucr.edu/~res/math153/history04Y.pdf

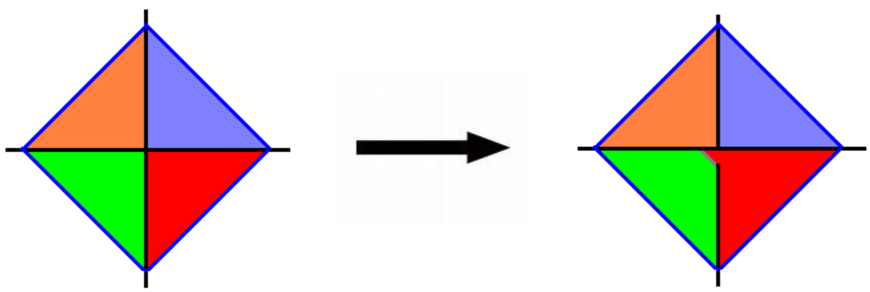
Appendix B —Euclidean geometry and modern mathematics

The types of results discussed in Appendix **A** are appealing in many respects, but few if any mathematicians would say they are mainstream topics. In fact, some feel that Euclidean geometry is a dead subject. Several comments on such views are given in the following document:

http://math.ucr.edu/~res/math153/history03e.pdf

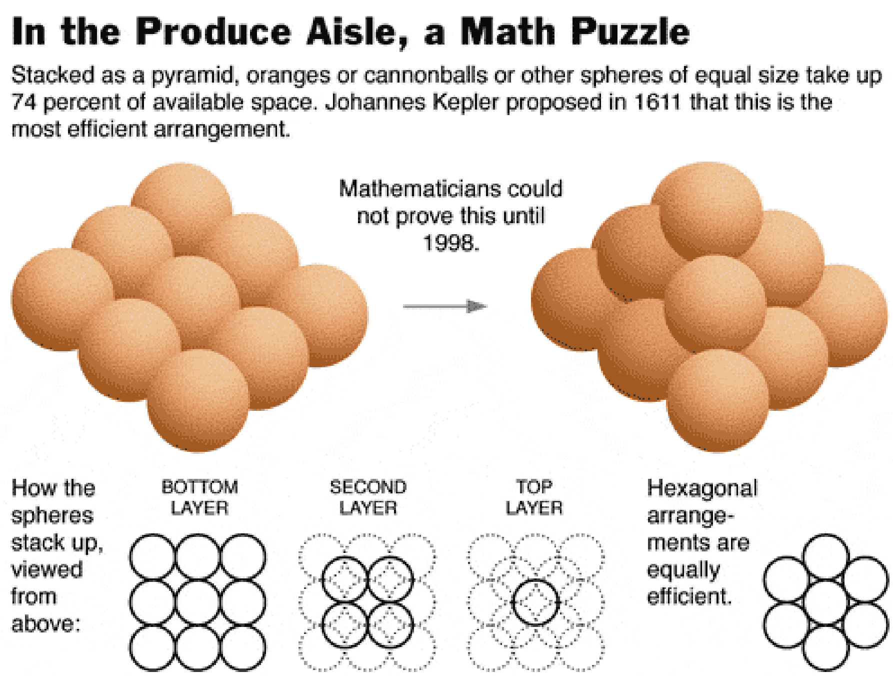
It is probably more accurate to say that problems from Euclidean geometry has still are of interest in mathematical research, but the focus has moved to more sophisticated types of questions. In particular, during the past half century mathematicians have solved two problems which are fairly simple to state but not at all easy to study:

1. The four color problem. The most intuitive formulation of this problem is that four colors suffice to color a "good" map on the plane (each country consists of a single connected piece, and no boundary point lies on the boundaries of more than three countries; in particular, this eliminates phenomena like the four corners point in the U. S. where Colorado, Utah, Arizona and New Mexico all meet — as in the picture below, one can always modify boundaries very slightly to achieve this regularity condition, and this can even be done more carefully without changing the areas of any of the regions).



Hands – on experience with map coloring suggests that one never needs more than four colors. This question was first studied at length in the 19^{th} century, and in 1890 P. J. Heawood (1861 – 1955) proved that five colors always suffice. The first correct proof that four colors always suffice was completed by K. Appel (1934 – 2013) and W. Haken (1928 –) in the middle of the nineteen seventies; their approach involved massive use of computers, generating major concerns about the validity of the argument, but subsequent proofs have successfully addressed these concerns. For further information on this topic, see the online article **http://en.wikipedia.org/wiki/Four_color_theorem**.

2. The Kepler sphere packing problem. Johannes Kepler (1571 – 1630) is best known for discovering the laws of planetary motion carrying his name, but he also made significant contributions to other parts of physics and mathematics. In particular, he is also known for stating his so – called ***Kepler sphere packing problem***, which was first formulated in 1611 and conjectures that ***the most efficient way of packing solid spheres into a box is the usual method*** in which oranges (or other solid round objects) are stacked on top of each other (see the figures below).



(Source: http://www.math.binghamton.edu/zaslav/Nytimes/+Science/+Math/sphere-packing.20040406.gif)

Although this conjecture intuitively seems very likely to be true and strong partial results were obtained over the years, a complete proof has been extremely elusive. In 1998 T. Hales (1958 –) announced a proof of this result. Many prominent experts in the area are confident that Hales has given a valid proof, but the argument depends on massive amounts of computer calculation, and current estimates are that it will take another decade or so before the accuracy of the computer calculations can be suitably verified. Some online references for further information are given below; the article from the ***New York Times*** is particularly informative (newspaper articles on advanced mathematical topics are extremely challenging to write and are often not highly reliable, but this one is an exception).

http://www.maa.org/devlin/devlin_9_98.html

http://www.sciencenews.org/sn_arc98/8_15_98/fob7.htm

http://mathworld.wolfram.com/KeplerConjecture.html

http://www.math.binghamton.edu/zaslav/Nytimes/+Science/+Math/sphere-packing.20040406.html