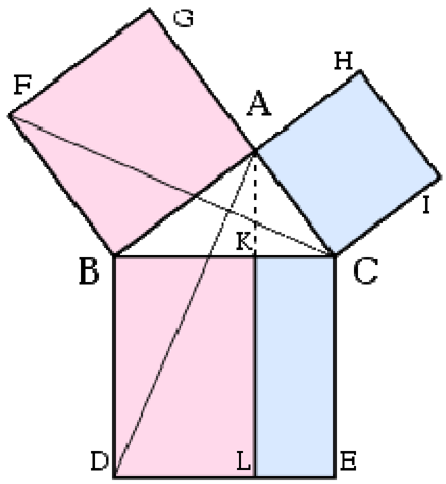
5. F. The Pappus Area Theorem

The final "Additional Exercise" for Unit 5 involves a remarkable, far – reaching generalization of the Pythagorean Theorem due to Pappus, which is sometimes called the ***Pappus Area Theorem***. In order to motivate the statement of Pappus' result, it is helpful to recall how Euclid proved the Pythagorean Theorem in the Elements: Given a right triangle **ABC** with a right angle at vertex **A,** he constructed three squares as below such that each edge of the triangle was also an edge to one of the squares, and he proved that the area of the region bounded by square **BCED** was equal to the sum of the areas bounded by the squares **ABFG** and **ACIH.**

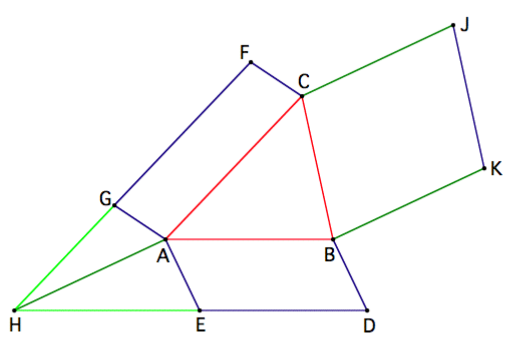


(Source: http://en.wikipedia.org/wiki/Pythagorean_theorem)

The Pappus Area Theorem generalizes to triangles which are not necessarily right triangles and parallelograms which are not necessarily squares but share sides with the triangle:

Theorem (*Pappus Area Theorem*). Let **ABC** be a triangle, and let parallelograms **ABDE** and **ACFG** be erected externally to **ABC** with respective bases **AB** and **AC.** Let **H** be the point where the lines **DE** and **FG** meet. If **BCJK** is the parallelogram erected upon **BC,** external to triangle **ABC** and having the sides **CJ** and **BK** parallel to and congruent with the segment **HA,** then the area of **BCJK** is the sum of the areas of **ABDE** and **ACFG.**

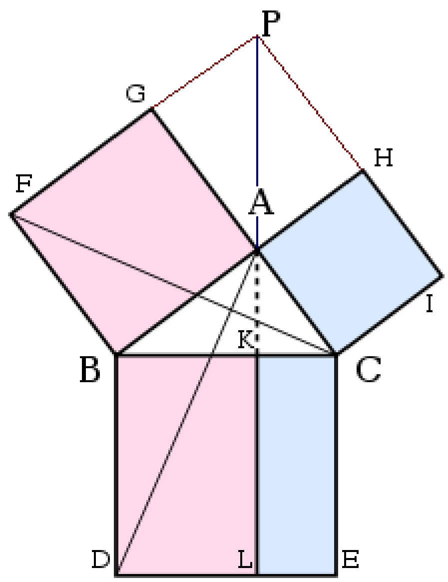
The geometric figures in the proof are summarized in the following drawing:



http://clem.mscd.edu/~talmanl/HTML/Pappus.html

This site also contains an animated description of the theorem (and many other illuminating animations are available from the related site **http://clem.mscd.edu/~talmanl/MathAnim.html**.

The relation between the Pappus Area Theorem and the Pythagorean Theorem is given by Additional Exercise 5 in **http://math.ucr.edu/~res/math153/math153exercises05.pdf**. Specifically, this exercise implies that, in the diagram below, the point **P** lies on the perpendicular from the right angle vertex **A** to the hypotenuse **BC.**



Further discussions of Pappus' generalization appear in the following sites:

http://jwilson.coe.uga.edu/emt725/Pappus/PappusAreas.html

http://jwilson.coe.uga.edu/EMT668/EMAT6680.2000/Burrell/Essay3/Essay3.html

In particular, the second contains an extremely well – illustrated proof of the Pappus Area Theorem.

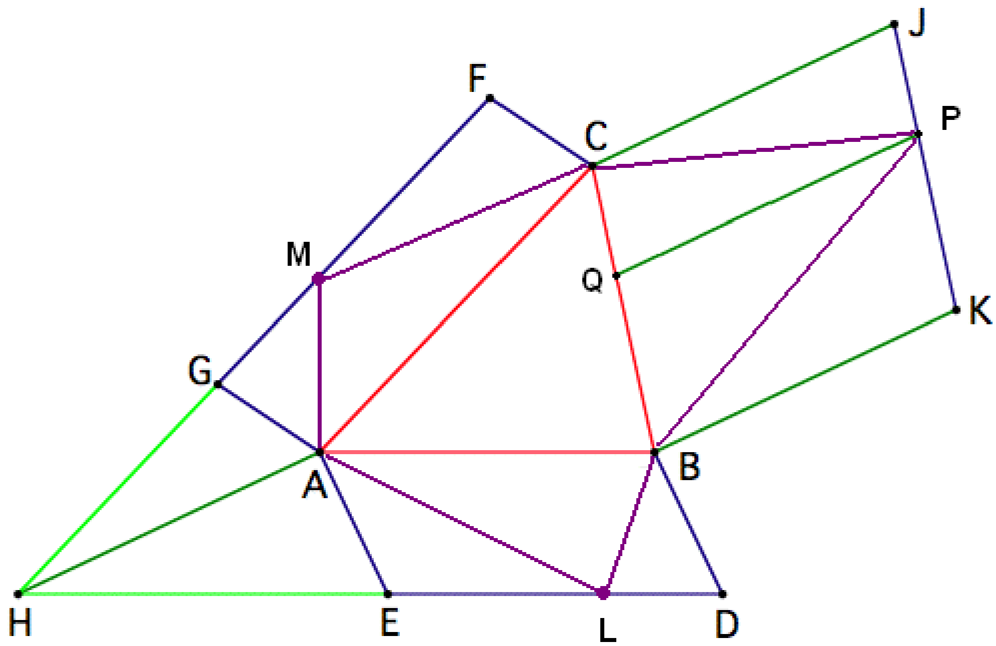
One more Additional Exercise for Unit 5. At the end of the file of exercises for Unit 5, we noted that one more additional exercise would be given in this document. Here it is:

6. Prove a generalization of the Pythagorean theorem similar to the Pappus Area Theorem which involves triangles instead of parallelograms. **[** ***Hint*****:** You need to formulate the correct statement first. The proof can be done by adapting the parallelogram proof. **]**

A solution to this exercise is sketched on the next page.

Solution to Additional Exercise 5.6

We want a setting in which there are triangles rather than parallelograms attached to the edges of the triangle. Let **L** and **M** be the third vertices of the triangles attached along the edges **[AB]** and **[AC]** respectively. Then we can construct two parallelograms which share these edges as before such that **L** lies on **DE** and **M** lies on **GH.**



With these choices of parallelograms, we obtain the same point **H** as before. Note that the areas of the regions enclosed by the triangles are half the areas of the regions enclosed by the parallelograms. Suppose now that we take an arbitrary point **Q** on the line **BC** and construct a line segment **[QP]** such that the lengths of **[HA]** and **[QP]** are equal, and **QP** is either equal or parallel to **HA.** If **Q** is not equal to **C,** then **P** cannot be equal to **J** (since each of **QP** and **CJ** is equal or parallel to **BK**). Furthermore, in this case we can say that **C, Q, P, J** form the vertices of a parallelogram (since there are parallel opposite sides of equal length). In either case, the point **P** must lie in the line **CJ** (if **Q = C** then **P = J,** while if **P** and **J** are different then they still lie on the unique parallel to **BC** through **J**). It follows that the area of the region bounded by triangle **BCP** is half the area of the region bounded by the parallelogram **BCJK.** Therefore we have

Area(BCP) = ½ **Area(BCJK)** =

½[Area(BCJK) + Area(BCJK)] = **Area(ALB) + Area(AMC)**

where the second equation holds by the Pappus Area Theorem, and the third holds as before since the areas of the regions enclosed by the given triangles are half the areas enclosed by the given parallelograms.