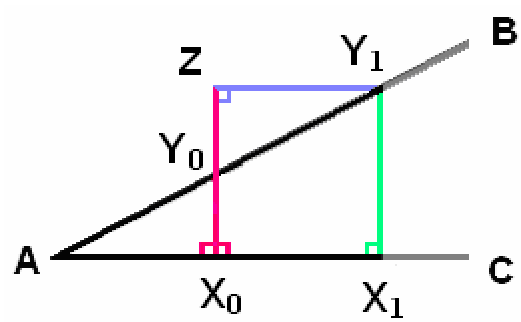
Aristotle's Axiom in neutral geometry

The following result receives a fair amount of attention in Greenberg:

<u>**Theorem**</u> <u>(Aristotle's Axiom)</u>. ***Assume that we are working inside a neutral plane.*** *Let* $\angle$**BAC** *be an acute angle, and let* k *be a positive real number. Then there are points* **X** *and* **Y** *on* **(AC** *and* **(AB** *respectively such that* **XY** *is perpendicular to* **AC** *and* d**(X, Y) >** k.

<u>Proof.</u> The idea is simple: We start out with any pair of points **X** and **Y** satisfying the all the conditions in the theorem except perhaps the inequality, and then we construct a new pair of points such **X** and **Y** that d**(U, V)** is at least twice d**(X, Y).** If this construction is repeated enough times, then the Archimedean Law implies that we shall obtain a pair of points for which the distance is greater than k.

Suppose that we are given points $\mathbf{X_0}$ and $\mathbf{Y_0}$ satisfying all the conditions in the theorem except possibly the inequality; since is acute and $\mathbf{Y_0}$ lies on **(AB,** it follows that the foot $\mathbf{X_0}$ of the perpendicular from $\mathbf{Y_0}$ to **AC** lies on the ray **(AC.** Choose $\mathbf{Y_1}$ on **(AB** such that $\mathbf{Y_0}$ is the midpoint of $\mathbf{[AY_1]}$; if $\mathbf{X_1}$ is the foot of the perpendicular from $\mathbf{Y_1}$ to **AC,** then as in the preceding sentence we know that $\mathbf{X_1}$ also lies on **(AC.** Furthermore, the midpoint condition implies the betweenness relationship $\mathbf{A{*}Y_0{*}Y_1}$, so that **A** and $\mathbf{Y_1}$ lie on opposite sides of the line $\mathbf{X_0Y_0}$. Since $\mathbf{X_0Y_0}$ and $\mathbf{X_1Y_1}$ have a common perpendicular, it follows that all points of the second line are on the same side of the first, and hence **A** and $\mathbf{X_1}$ lie on oppositie sides of $\mathbf{X_0Y_0}$, so that we have $\mathbf{A{*}X_0{*}X_1}$.



Now choose **Z** so that $\mathbf{Y_0}$ is the midpoint of **[XZ].** Then by SAS we have $\triangle\mathbf{AY_0X_0} \cong \triangle\mathbf{Y_1Y_0Z}$; therefore, it follows that $\mathbf{Y_1Z}$ is perpendicular to $\mathbf{X_0Z} = \mathbf{X_0Y_0}$. If we combine all the perpendicularity conditions, we see that $\mathbf{X_0}$, $\mathbf{X_1}$, $\mathbf{Y_1}$, **Z** form the vertices of a Lambert quadrilateral with right angles at all vertices except possibly $\mathbf{Y_1}$. Therefore Exercise **V.3.9** implies that $d(\mathbf{X_0}, \mathbf{Z}) \leq d(\mathbf{X_1}, \mathbf{Y_1})$ [***<u>Note:</u>*** The statement of that exercise should be corrected to state that there are right angles at **A, B** and **D.**]. By construction we also know that $d(\mathbf{X_0}, \mathbf{Z}) = 2d(\mathbf{X_0}, \mathbf{Y_0})$, and hence it also follows that $d(\mathbf{X_0}, \mathbf{Y_0}) \leq \frac{1}{2}d(\mathbf{X_1}, \mathbf{Y_1})$; as indicated above, this suffices to complete the argument.■