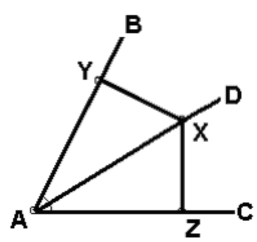
MORE PROOFS IN NEUTRAL GEOMETRY

We shall indicate how some proofs in the course notes can be modified so that they are valid in neutral geometry.

<u>Theorem III.4.6.</u> *Let* **A, B** *and* **C** *be noncollinear, and let* **[AD** *be the bisector of* $\angle$**BAC.** *Given a point* **X** *in* **Int** $\angle$**BAC,** *let* $\mathbf{Y_X}$ *and* $\mathbf{Z_X}$ *be the feet of the perpendiculars from* **X** *to* **AB** *and* **AC** *respectively. Then* $\mathbf{X} \in$ **(AD** *if and only if* $d(\mathbf{X}, \mathbf{Y_X}) = d(\mathbf{X}, \mathbf{Z_X})$.

<u>Proof.</u> We shall reproduce the argument given in the notes, crossing out the passages which rely on Playfair's Postulate and highlighting their replacements or other insertions. One can check directly that the proof of Lemma **III.4.7** in the notes is valid in neutral geometry, so the use of this result in the modified argument will not create any problems.

***<u>Suppose first that</u>* X *<u>lies on the bisector</u>*.** Since $|\angle \mathbf{BAC}|$ is less than $\mathbf{180°}$, it follows that both $|\angle \mathbf{XAB}|$ and $|\angle \mathbf{XAC}|$ are less than $\mathbf{90°}$, so by the lemma **(III.4.7)** we know that **Y** lies on **(AB** and also **Z** lies on **(AC.**

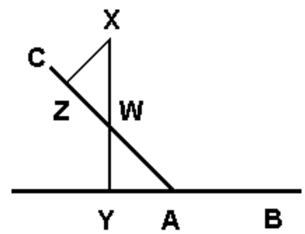


Since $|\angle \mathbf{XZA}| = |\angle \mathbf{XYA}| = \mathbf{90°}$ and $|\angle \mathbf{XAZ}| = |\angle \mathbf{YAZ}| = ½|\angle \mathbf{BAC}|$, we have $\triangle \mathbf{ZAX} \cong \triangle \mathbf{YAX}$ by **AAS,** and hence $d(\mathbf{X},\mathbf{Y}) = d(\mathbf{X}, \mathbf{Z})$.

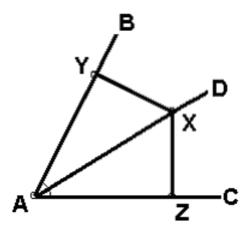
<u>Conversely, suppose that</u> $\mathbf{X} \in$ **Int** $\angle$**BAC** ***<u>and</u>*** $d(\mathbf{X},\mathbf{Y}) = d(\mathbf{X},\mathbf{Z})$. We claim that **Y** and **Z** lie on the open rays **(AB** and **(AC** respectively. Since $|\angle \mathbf{XAB}| + |\angle \mathbf{XAC}| = |\angle \mathbf{BAC}| < \mathbf{180°}$ it follows that at least one of the terms on the left hand side must be strictly less than $\mathbf{90°}$. Without loss of generality, we might as well assume that $|\angle \mathbf{XAC}| < \mathbf{90°}$; if not, we can retrieve the result when $|\angle \mathbf{XAB}| < \mathbf{90°}$ by reversing the roles of **B** and **C** and of **Y** and **Z** in the argument that follows. By the lemma **(III.4.7)** the condition $|\angle \mathbf{XAC}| < \mathbf{90°}$ implies that **Z** lies on **(AC.** If **Y** does not lie on **(AB,** then as in the lemma we either have $\mathbf{Y} = \mathbf{A}$ or else $\mathbf{Y*A*B}$. We can dispose of the case $\mathbf{Y} = \mathbf{A}$ as follows: If this happens then we have a right triangle $\triangle \mathbf{XZA}$, and since the hypotenuse is strictly longer than either of the other sides (if $\triangle \mathbf{FGH}$ is right triangle with a right angle at **G,** then Corollary **III.2.2** implies that $\angle$**FHG** is acute and Theorem **III.2.5** implies that $d(\mathbf{F},\mathbf{H}) > d(\mathbf{F},\mathbf{G})$) this means that $d(\mathbf{X},\mathbf{Y}) = d(\mathbf{X},\mathbf{A}) > d(\mathbf{X},\mathbf{Z})$, contradicting our assumption that $d(\mathbf{X},\mathbf{Y}) = d(\mathbf{X},\mathbf{Z})$.

Thus it remains to eliminate the possibility that $\mathbf{Y*A*B}$ holds. However, if $\mathbf{Y*A*B}$ holds, then **Y** and **B** lie on opposite sides of **AC.** Since **B** and **X** lie on the same side of **AC** by hypothesis, it follows that **Y** and **X** lie on opposite sides of **AC.** Thus the line **AC** and the segment **(XY)** have some point **W** in common. It follows that $d(\mathbf{X}, \mathbf{Y}) > d(\mathbf{X},\mathbf{W})$. Also, since **XZ** is perpendicular to **AC** and meets the latter at **Z,** it follows (~~say,~~ from ~~the~~

~~Pythagorean Theorem~~ argument at the end of this sentence) that $d(\mathbf{X},\mathbf{W}) \geq d(\mathbf{X},\mathbf{Z})$; if **W** = **Z** this is immediate, while if $\mathbf{W} \neq \mathbf{Z}$ then $\triangle\mathbf{XZW}$ is right triangle with a right angle at **Z,** so that Corollary **III.2.2** implies that $\angle\mathbf{XWZ}$ is acute and Theorem **III.2.5** implies that $d(\mathbf{X},\mathbf{W}) > d(\mathbf{X},\mathbf{Z})$. Combining the observations in the preceding sentences, we have $d(\mathbf{X},\mathbf{Y}) > d(\mathbf{X},\mathbf{Z})$, contradicting our assumption that these were equal. Therefore $\mathbf{Y}*\mathbf{A}*\mathbf{B}$ is also impossible, and the only remaining option is for **Y** to lie on **(AB.**



Now that we know that **Y** and **Z** lie on the open rays **(AB** and **(AC** respectively, the rest of the proof is straightforward. Triangles $\triangle\mathbf{XYA}$ and $\triangle\mathbf{XZA}$ are right triangles with right angles at **Y** and **Z** respectively (for the sake of convenience, we have inserted a copy of the first drawing in the proof).



We know that $d(\mathbf{X},\mathbf{A}) = d(\mathbf{X},\mathbf{A})$ and also $d(\mathbf{X},\mathbf{Y}) = d(\mathbf{X},\mathbf{Z})$, so by the ~~Pythagorean Theorem~~ Hypotenuse – Side Congruence Theorem for right triangles we have ~~we also know that $d(\mathbf{A},\mathbf{Y}) = d(\mathbf{A},\mathbf{Z})$. Therefore~~ $\triangle\mathbf{XYA} \cong \triangle\mathbf{XZA}$, ~~by **SSS**,~~ so that $|\angle\mathbf{XAY}| = |\angle\mathbf{XAZ}|$. Since **Y** and **Z** lie on the open rays **(AB** and **(AC** respectively, we have $\angle\mathbf{XAB} = \angle\mathbf{XAY}$ and $\angle\mathbf{XAZ} = \angle\mathbf{XAC}$. By assumption **X** lies in the interior of $\angle\mathbf{BAC}$, and therefore by the Additivity Postulate we have $|\angle\mathbf{BAC}| = |\angle\mathbf{BAX}| + |\angle\mathbf{XAC}| = 2|\angle\mathbf{BAX}| = 2|\angle\mathbf{XAC}|$, so that

$$|\angle\mathbf{BAX}| = |\angle\mathbf{XAC}| = \tfrac{1}{2}|\angle\mathbf{BAC}|,$$

which means the ray **[AX** is the bisector of $\angle\mathbf{BAC}$.■

Once we have the modified argument given above, the proof in the notes for the following theorem is also valid in neutral geometry:

Theorem III.4.8. *Given* $\triangle\mathbf{ABC}$, *let* **[AD, [BE** and **[CF** *be the bisectors of* $\angle\mathbf{BAC}$, $\angle\mathbf{ABC}$ *and* $\angle\mathbf{BCA}$ *respectively. Then the lines* **AD, BE** *and* **CF** *have a point in common, and it lies in the interior of* $\triangle\mathbf{ABC}$.■

REMARK. In contrast, the theorem on excenters (Theorem **III.4.8**) does **NOT** hold in a neutral plane for which Playfair's Postulate is false; however, we shall not attempt to explain or prove this assertion here.

The Triangle Midpoint Theorem in hyperbolic geometry

In Euclidean geometry, if we are given $\triangle$**ABC** such that **D** and **E** are the midpoints of **[AB]** and **[AC]** respectively, then we can conclude that **DE** || **BC** and d**(D, E)** $=$ ½d**(B, C).** In hyperbolic geometry, one can still prove that **DE** || **BC;** more precisely, in hyperbolic geometry one can prove that these two lines have a common perpendicular. On the other hand, in hyperbolic geometry we have d**(D, E)** $<$ ½d**(B, C).** Further information on this appears in Exercise 2 on pages $269-270$ of Greenberg.