THE CLASSICAL APPROACH TO SIMILARITY

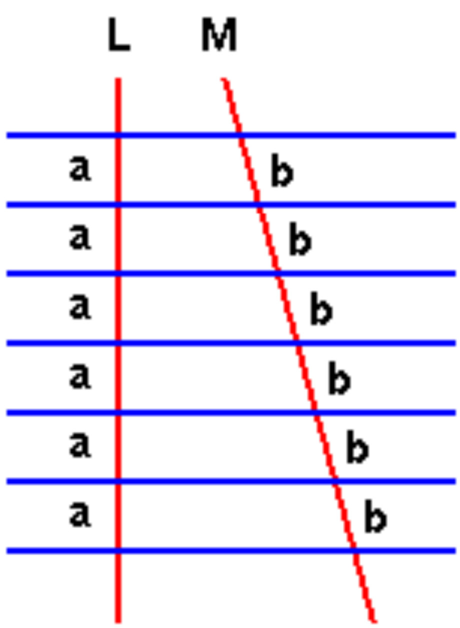
In the lectures we used geometric transformations as a basis for studying similarity theory. One crucial point in our approach was the existence of similarity transformations whose effect on distances are multiplication by an arbitrary positive real number k. Since the classical approach in Greek geometry did not have a mathematically rigorous notion of geometrical transformation, clearly it needed alternatives. It took a couple of centuries for the Greeks to develop tools which justified similarity theory for all possible ratios of interest, and we shall sketch them here.

Before discoveries of Eudoxus in the 4^{th} century B.C.E., the Greek geometers were only able to prove similarity theorems for triangles in the ***commensurable*** case. For a pair of triangles $\triangle$ **ABC** and $\triangle$ **DEF,** this means that the common ratio of the lengths of the corresponding sides

$$|AB|/|DE| \;=\; |AB|/|DE| \;=\; |AB|/|DE|$$

is a ***rational number.*** Greek geometers were able to attack the similarity problem effectively using the following fact, which one might call the ***Notebook Paper Theorem.***

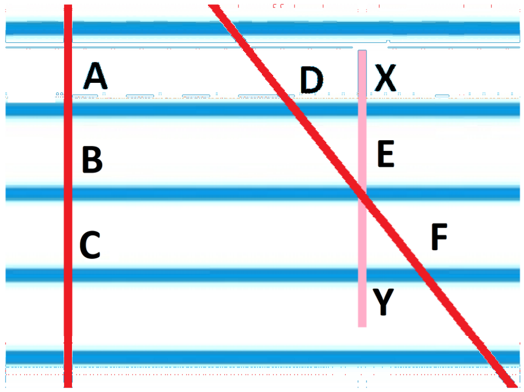
Suppose that we are given a family of distinct parallel lines $P_1, P_2, \ldots$ *and two transversals* **L** *and* **M** *that are neither parallel or identical to these lines. Assume that the points satisfy the ordering condition* $P_1 * P_2 * \ldots * P_n$. *For each* k *let* A_k *be the point at which* **L** *and* P_k *intersect, and let* B_k *be the point at which* **M** *and* P_k *intersect. If all the segment lengths* $|A_k A_{k+1}|$ *are equal to* a, *for some fixed* a, *then all the segment lengths* $|B_k B_{k+1}|$ *are equal to* b, *for some fixed* b.



One can prove this by induction on the number of parallel lines. In fact the initial case with three parallel lines and the inductive hypothesis

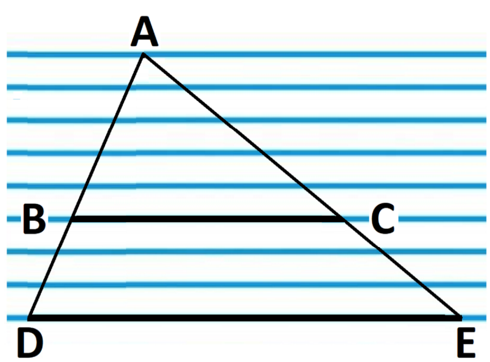
(*true for* n *parallel lines*) **implies** (*true for* $n+1$ *parallel lines*).

Both reduce to analyzing the situation in the drawing below; the goal is to prove that **|AB|** = **|BC|** implies that **|DE|** = **|EF|.**



In this drawing the line **XY** is parallel to **AB,** and one also has the three betweenness conditions suggested in the drawing. By construction we have parallelograms **ABEX** and **BCYE**, so that **|XE|** = **|AB|** = **|BC|** = **|YE|.** This leads to concluding that $\triangle$ **XED** is congruent to $\triangle$ **YEF** by **A.S.A.** (there are alternate interior angles and **X** and **Y,** and vertical angles at **E**), and hence that **|DE|** = **|EF|.**

A typical example of two triangles with commensurable ratios is given below. In this case we have **|AB|** = $5p$ and **|AD|** = $8p$ for some positive real number p. By the Notebook Paper Theorem we then have **|AB|** = $5q$ and **|AD|** = $8q$ for some positive real number q.



Similar considerations hold if $5{:}8$ is replaced by an arbitrary ratio $M{:}N$ where M and N are arbitrary positive integers. All this was presumably known very early in the development of Greek geometry. However, when the Pythagoreans discovered that the square root of 2 is irrational, it was immediately clear that the argument for commensurable quantities only yielded a partial result on proportionality.

Euclid's ***Elements*** gave a satisfactory treatment for incommensurable ratios using ideas developed by Eudoxus a few decades earlier. Here is a formal statement of Eudoxus' criterion for two ratios to be equal:

Two ratios of (positive real) numbers a/b *and* c/d *are equal if and only if for each pair of positive integers* m *and* n *we have the following***:**

$$ma < nb \quad \textit{implies} \quad mc < nd$$
$$ma > nb \quad \textit{implies} \quad mc > nd$$

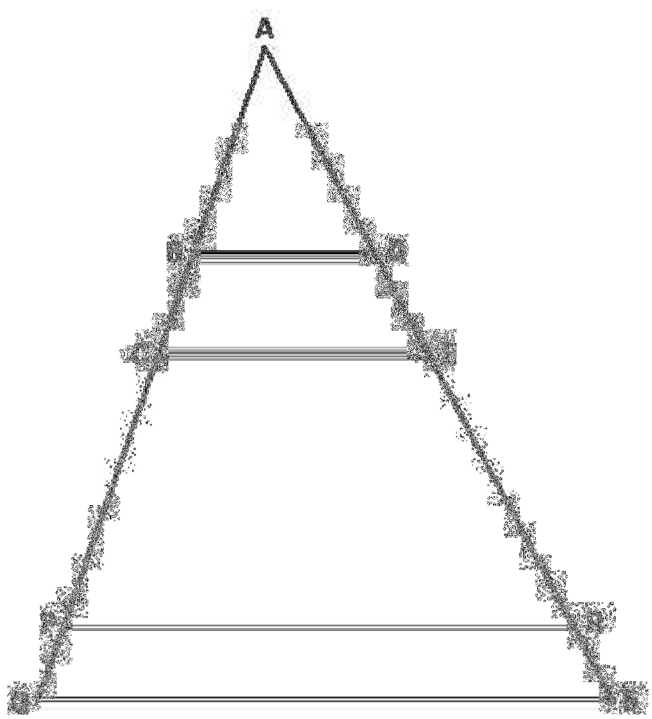
The derivation of this criterion is based upon a fundamentally important **<u>rational density property</u>** of the real numbers**:**

If we are given real numbers x *and* y *such that* $x < y$, *then there is a rational number* r *such that* $x < r < y$.

A derivation of Eudoxus' condition from the rational density property is given in the document **http://www.math.ucr.edu/~res/math153/history03a.pdf**.

Application of the Condition of Eudoxus to proportionality questions

Suppose now that we have triangles $\triangle$**ABD** and $\triangle$**ACE** as in the figure below**,** where **BD** is parallel to **CE;** as in the figure we assume that the rays **[AB** and **[AC** are the same and likewise that the rays **[AD** and **[AE** are the same**.** Let a = |**AB**|**,** b = |**AC**|**,** c = |**AD**| and d = |**AE**|**.** We want to use the Condition of Eudoxus to conclude that $a/b = c/d$.



Suppose first that m and n are positive integers such that $ma < nb$. We want to show that $mc < nd$. We can find points **P** and **Q** on the ray **[AB** = **[AC** such that |**AP**| = ma and

|AQ| $=$ nb. Since $ma < nb$, it follows that **P** is between **A** and **Q.** One can then find unique parallel lines to **BD** and **CE** through **P** and **Q.** These two lines will meet the line **AD** $=$ **AE** in two points **R** and **S.** In fact, Pasch's Theorem will imply that **S** and **R** also lie on the ray **[AD** $=$ **[AE** and that **R** is between **A** and **S.**

The proportionality results in the commensurable case now imply that

$$|\mathbf{AR}|/|\mathbf{AD}| = m = |\mathbf{AP}|/|\mathbf{AB}| \quad \text{and}$$

$$|\mathbf{AS}|/|\mathbf{AE}| = n = |\mathbf{AQ}|/|\mathbf{AC}|.$$

Therefore **|AR|** $=$ mc and **|AS|** $=$ nd also hold. By observations from the previous paragraph we know that **|AR|** $<$ **|AS|,** and thus we may use the preceding sentences to rewrite this as $mc < nd$. To summarize, we have now shown that $ma < nb$ implies $mc < nd$.

If we have $ma > nb$, then we may proceed similarly. The argument is basically the same except that **Q** will be between **A** and **P,** and this in turn will imply that **S** is between **A** and **R.** Following the same line of reasoning in this case, one concludes that $ma > nb$ implies $mc > nd$. Therefore we have established both parts of the Condition of Eudoxus, and consequently we have shown that $a/b = c/d$; by definition of the numbers a, b, c, d in this equation, the desired proportionality equation **|AB|/|AC|** $=$ **|AD|/|AE|** is an immediate consequence.