Notes on the proof of Theorem III.4.6

The proofs of this result in many geometry texts often overlook a crucial point: *If* $X \in \mathrm{Int}\ \angle BAC$ *satisfies either of the conditions*

(i) *X lies on the angle bisector,*

(ii) *the distances from X to the lines AB and AC are equal,*

then the feet of the perpendiculars from X to the lines AB and AC lie on the open rays $(AB$ and $(AC$ respectively.

The first possibility is dealt with in Lemma III.4.7, and the second possibility is dealt with in the converse portion of the theorem's proof beginning with the final paragraph beginning on page 125 (page 5 of the file). The basic idea is to start with X in the interior and notice that at least one of the angles $\angle CAD$ and $\angle DAB$ must be acute (their measures add up to something less than 180°). We can use Lemma III.4.7 to handle acute angles, so we are left to consider right and obtuse angles. The argument beginning with the first full paragraph of page 126 (page 6 of the file) considers the hypothetical possibility that an angle is acute; it derives a contradiction by showing that the distances from X to the two lines must be unequal. Thus the feet of both perpendiculars must lie on the open rays if the two distances are equal.∎