CHAPTER

17

Cartesian Coordinate Systems

Obviously, all readers of this book know about coordinate systems, from elementary analytic geometry. For the sake of completeness, however, we explain them here from the beginning. To achieve speed and simplicity, and reduce the amount of outright repetition, we have introduced various novelties in the derivations.

In a plane E we set up a Cartesian coordinate system in the following way. First we choose a line X, with a coordinate system as given by the ruler postulate. The zero point of X will be called the *origin*. We now take a line Y, perpendicular to X at the origin, with a coordinate system in which the origin has coordinate $= 0$.

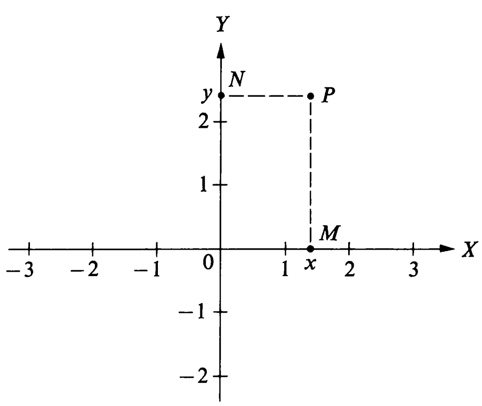


Figure 17.1

Given a point P of E, we drop a perpendicular to a point M of X. The coordinate x of M on X is called the *x-coordinate*, or the *abscissa*, of P. We drop a perpendicular from P to a point N of Y. The coordinate of N on Y is called the *y-coordinate,* or the *ordinate,* of P. Thus to every point P of E there corresponds an ordered pair (x,y) of real numbers, that is, an element of the product set $\mathbb{R} \times \mathbb{R}$. Clearly this is a one-to-one correspondence

$$E \leftrightarrow \mathbb{R} \times \mathbb{R}.$$

For short, we shall speak of "the point (x,y)," meaning, of course, the point corresponding to (x,y) in the coordinate system under discussion.

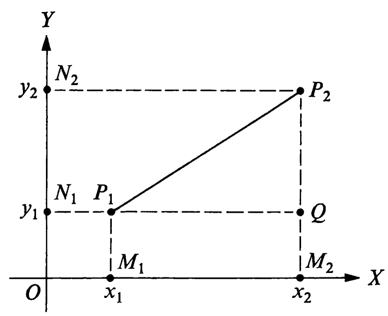


Figure 17.2

■ ***THEOREM 1.*** The distance between the points $P_1 = (x_1,y_1)$ and $P_2 = (x_2,y_2)$ is given by the formula

$$P_1P_2 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

PROOF. Let M_1, N_1, M_2, N_2 be the projections of P_1 and P_2 onto the axes, as in the definition of coordinates. If $x_1 = x_2$, then

$$\overleftrightarrow{P_1P_2} \parallel \overleftrightarrow{N_1N_2}, \qquad |x_2 - x_1| = 0,$$

and

$$\begin{aligned} P_1P_2 &= |y_2 - y_1| \\ &= \sqrt{(y_2 - y_1)^2} \\ &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}. \end{aligned}$$

(Here we are ignoring the trivial case where $P_1 = N_1$ and $P_2 = N_2$.) If $y_1 = y_2$, the same conclusion follows in a similar way. Suppose, then, that $x_1 \neq x_2$ and $y_1 \neq y_2$, as in the figure. Then the horizontal line through P_1 intersects the vertical line through P_2, in a point Q, and $\triangle P_1P_2Q$ has a right angle at Q. (Here, and hereafter, a horizontal line is X or a line parallel to X; and a vertical line is Y or a line parallel to Y.) Thus

$$P_1Q = M_1M_2,$$

and

$$P_2Q = N_2N_1,$$

either because the point pairs are the same or because opposite sides of a rectangle are congruent. By the Pythagorean theorem,

$$P_1P_2^2 = P_1Q^2 + P_2Q^2.$$

Therefore

$$\begin{aligned} P_1P_2^2 &= M_1M_2^2 + N_1N_2^2 \\ &= |x_2 - x_1|^2 + |y_2 - y_1|^2 \\ &= (x_2 - x_1)^2 + (y_2 - y_1)^2, \end{aligned}$$

and from this the distance formula follows. □

By a *linear equation in x and y* we mean an equation of the form

$$Ax + By + C = 0,$$

where A, B, and C are real numbers, and A and B are not both $= 0$. By the *graph* of an equation, we mean the set of all points that satisfy the equation. More generally, by the graph of a *condition* we mean the set of all points that satisfy the given condition. Thus the interior of a circle with center Q and radius r is the *graph* of the condition $PQ < r$; and one of our theorems tells us that the perpendicular bisector of a segment $\overline{AB}$ is the *graph* of the condition $PA = PB$.

■ ***THEOREM 2.*** Every line in E is the graph of a linear equation in x and y.

PROOF. Let L be a line in E. Then L is the perpendicular bisector of some segment $\overline{P_1P_2}$, where $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$. Thus L is the graph of the condition

$$PP_1 = PP_2.$$

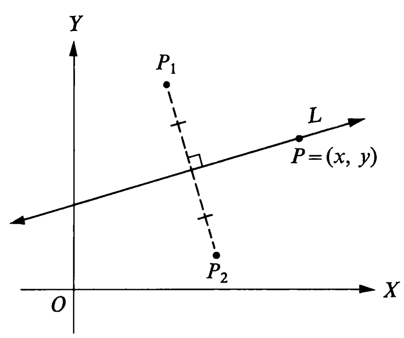


Figure 17.3

With $P = (x, y)$ this can be written algebraically in the form

$$\sqrt{(x - x_1)^2 + (y - y_1)^2} = \sqrt{(x - x_2)^2 + (y - y_2)^2},$$

or

$$x^2 - 2x_1x + x_1^2 + y^2 - 2y_1y + y_1^2 = x^2 - 2x_2x + x_2^2 + y^2 - 2y_2y + y_2^2$$

or

$$2(x_2 - x_1)x + 2(y_2 - y_1)y + (x_2^2 + y_2^2 + x_1^2 + y_1^2) = 0.$$

This has the form

$$Ax + By + C = 0.$$

And A and B cannot both be $= 0$, because then we would have $x_2 = x_1$ and $y_2 = y_1$; this is impossible, because $P_1 \neq P_2$. □

■ ***THEOREM 3.*** If L is not vertical, then L is the graph of an equation of the form

$$y = mx + k.$$

PROOF. L is the graph of an equation

$$Ax + By + C = 0.$$

Here $B \neq 0$, because for $B = 0$ the equation takes the form $x = -C/A$; and the graph is then vertical. Therefore we can divide by B, getting the equivalent equation

$$y = -\frac{Ax}{B} - \frac{C}{B}.$$

This has the desired form, with

$$m = -\frac{A}{B}, \qquad k = -\frac{C}{B}. \quad \square$$

■ ***THEOREM 4.*** If L is the graph of $y = mx + b$, and (x_1,y_1), (x_2,y_2) are any two points of L, then

$$\frac{y_2 - y_1}{x_2 - x_1} = m.$$

PROOF. Since both points are on the line, we have

$$y_2 = mx_2 + k, \qquad y_1 = mx_1 + k.$$

Therefore

$$y_2 - y_1 = m(x_2 - x_1),$$

and $x_2 \neq x_1$, because L is not vertical. Therefore

$$\frac{y_2 - y_1}{x_2 - x_1} = m. \quad \square$$

Thus the number m is uniquely determined by the line. It is called the *slope* of the line.

■ ***THEOREM 5.*** Let L and L' be two nonvertical lines, with slopes m and m'. If L and L' are perpendicular, then

$$m' = -\frac{1}{m}.$$

PROOF. Let

$$P_1 = (x_1, y_1) \quad \text{and} \quad P_2 = (x_2, y_2)$$

be points of L', such that L is the perpendicular bisector of $\overline{P_1 P_2}$. (See Fig. 17.3.) As in the proof of Theorem 2, L is the graph of the equation

$$2(x_2 - x_1)x + 2(y_2 - y_1)y + (x_2^2 + y_2^2 + x_1^2 + y_1^2) = 0.$$

This has the form

$$Ax + By + C = 0,$$

where

$$A = 2(x_2 - x_1), \qquad B = 2(y_2 - y_1).$$

Therefore

$$m = -\frac{A}{B} = -\frac{2(x_2 - x_1)}{2(y_2 - y_1)} = -\frac{x_2 - x_1}{y_2 - y_1}.$$

But, by Theorem 4, we have

$$m' = \frac{y_2 - y_1}{x_2 - x_1}.$$

Therefore $m' = -(1/m)$, which was to be proved. $\square$

■ ***THEOREM 6.*** Every circle is the graph of an equation of the form

$$x^2 + y^2 + Ax + By + C = 0.$$

PROOF. By the distance formula, the circle with center (a, b) and radius r is the graph of the equation

$$\sqrt{(x - a)^2 + (y - b)^2} = r,$$

or

$$x^2 - 2ax + a^2 + y^2 - 2by + b^2 - r^2 = 0.$$

This has the required form, with

$$A = -2a,$$
$$B = -2b,$$
$$C = a^2 + b^2 - r^2. \quad \square$$

The converse of Theorem 6 is false, of course. The graph of

$$x^2 + y^2 = 0$$

is a point, and the graph of

$$x^2 + y^2 + 1 = 0$$

is the empty set.

Problem Set

In proving the following theorems, try to use as little geometry as possible, putting the main burden on the algebra and on the theorems of this section.

1. Show that the graph of an equation of the form

$$x^2 + y^2 + Ax + By + C = 0$$

is always a circle, a point, or the empty set.

2. Show that if the graphs of the equations

$$y = m_1x + k_1, \qquad y = m_2x + k_2$$

are two (different) intersecting lines, then $m_1 \neq m_2$.

3. Show that if $m_1 = m_2$, then the graphs are either parallel or identical.

4. In the chapter on similarity, we defined

$$A_1, B_1, C_1 \sim A_2, B_2, C_2$$

to mean that all the numbers in question were positive and that

$$\frac{A_2}{A_1} = \frac{B_2}{B_1} = \frac{C_2}{C_1}.$$

Let us generalize this in the following way. Given A_1, B_1, C_1, not all $= 0$. If there is a $k \neq 0$ such that

$$A_2 = kA_1, \qquad B_2 = kB_1, \qquad C_2 = kC_1,$$

then we say that the sequences A_1, B_1, C_1 and A_2, B_2, C_2 are *proportional,* and we write

$$A_1, B_1, C_1 \sim A_2, B_2, C_2.$$

With this understanding, show that if

$$A_1x + B_1y + C_1 = 0 \quad \text{and} \quad A_2x + B_2y + C_2 = 0$$

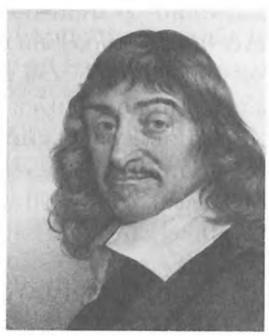
have the same line L as their graph, then

$$A_1, B_1, C_1 \sim A_2, B_2, C_2.$$

[*Hint:* Discuss first the case where L is vertical, and then the case where L is not vertical.]

5. Describe the graphs of the following equations.
(a) $x^2 + y^2 + 1 + 2x + 2y + 2xy = 0$
(b) $xy = 0$
(c) $x^3 + xy^2 - x = 0$

René Descartes
(1596–1650)



Descartes is a famous man in two quite separate domains: he is known among philosophers as a great philosopher and among mathematicians as a great mathematician.

His greatest contribution to mathematics was the discovery of coordinate systems and their application to problems of geometry. Ever since then algebra and geometry have worked together, to the advantage of both. To this day, coordinate systems of the sort used in this book are referred to as Cartesian coordinate systems, in honor of their inventor. The concept of coordinates was the first really fundamental contribution to geometry after the Greeks. (The word *Cartesian* comes from *Cartesius,* which is the Latin form of Descartes' name.)

Part of the credit for Descartes' discovery should go to Pierre Fermat, who had much the same ideas at about the same time. Fermat was one of the few great amateur mathematicians. He worked for the French government, and pursued mathematics in his spare time. He wrote letters to his friends about his discoveries, and never published them in any other form. But the material in Fermat's letters is now included in all the standard books on the theory of numbers.

The development of coordinate systems laid the foundation for the development of calculus, soon thereafter, by Newton and Leibniz. Thus Descartes must have been one of the men Newton had in mind when he said that he had stood on the shoulders of giants.

In otherwise well-informed circles, we often encounter the notion that Cartesian coordinate systems can be used to solve or to avoid problems in the foundations of geometry. We also encounter the notion that Descartes invented coordinate systems for this reason. These notions are based on misunderstandings of mathematics and of its history, respectively.

First, to set up a coordinate system, we need to know quite a lot about geometry. For example, if we do not know that the perpendicular to a line, through a given point, exists and is unique, we cannot explain what we mean by the x-coordinate of a point. By the time we can do this, the whole issue of the foundations is over, for better or worse, usually worse.

Second, Descartes invented coordinate systems to solve problems that he could not solve in any other way. In his time, nobody was worried about the foundations of geometry. Euclid was still regarded as a model of deductive rigor. What everybody was worried about was the real number system. Then, mathematicians (writing in Latin) called the negative numbers the *numeri ficti,* that is, the fictitious numbers, the numbers that are not really there. The situation was awkward: mathematicians were getting right answers to difficult algebraic problems by methods that they felt sheepish about. The scheme worked like this:

1. Pretend that negative real numbers exist (though you know very well that they do not). This gives a system $\mathbb{R}$ in which half the numbers are "numeri ficti." In the new system, to every x there corresponds a number $-x$ such that $x + (-x) = 0$.
2. Hopefully assume that the laws that govern the positive numbers also hold in the new system. Then $ab = ba$, $a(b + c) = ab + ac$, and so on.
3. Assume that $(-a)b = -(ab)$ and $(-a)(-b) = ab$, always.
4. Postpone, to some later century, the problem of justifying these procedures.

The real number system seems simple to us, because to us, the preceding laws are habits; but the laws were not habits to the people who invented the system, and so, to them, the system was mysterious. For example, if we are asked, what is $(-2)(-3)$, we "know" that the answer is 6, but what on earth was the meaning of the question?

The foundations of analysis were straightened out in the nineteenth century, when Descartes had long been dead. But in his time, the real number system was so shaky that nobody dreamed of using it as a foundation for anything else.