Math 133
Fall 2013

STILL MORE PROOFS IN NEUTRAL GEOMETRY

The previous two documents in this series summarize many results from classical synthetic geometry which can be proved without using Playfair's Postulate or some equivalent statement. In this note we shall take things one step further and discuss a few basic results about circles (from Section III.6 of the notes) which are also valid in neutral geometry. The most fundamental examples are the Line-Circle Theorem (Theorem III.6.1) and the Two Circle or Circle-Circle Theorem (Theorem III.6.3).

In most cases, the proofs of theorems about circles in neutral geometry rely heavily on basic properties of continuous functions on spaces which have an abstract notion of distance (*i.e.*, **metric spaces**), and consequently they are at a slightly higher level than most of this course. We shall use the following text as our source for further information on metric spaces:

> W. A. Sutherland, *Introduction to Metric and Topological Spaces* (Second Ed.), Oxford University Press, New York, 2009.

Definitions and geometric examples

The notion of distance between two points in $\mathbb{R}^n$ is fundamentally important in multivariable calculus, some aspects of linear algebra, and geometry as developed in this course. It turns out that an extremely short list of properties for distances are enough to prove abstract versions of many important results from advanced calculus and real variables courses.

Definition. A *metric space* is a pair $(X, \mathbf{d})$ consisting of a set X and a function $\mathbf{d} : X \times X \to \mathbb{R}$ (sometimes called the **metric** or **distance function**, with $\mathbf{d}(x, y)$ being called the **distance** from x to y, or between x and y) such that the following properties hold:

(MS1) $\mathbf{d}(x, y) \geq 0$ for all $x, y \in X$.
(MS2) $\mathbf{d}(x, y) = 0$ if and only if $x = y$.
(MS3) $\mathbf{d}(x, y) = \mathbf{d}(y, x) \geq 0$ for all $x, y \in X$.
(MS4) $\mathbf{d}(x, z) \leq \mathbf{d}(x, y) + \mathbf{d}(y, z)$ for all $x, y, z \in X$.

The last property is often called the *Triangle Inequality.* Our first observation is more or less predictable:

PROPOSITION 0. *The distance function in a neutral plane satisfies the defining properties for a metric space.*

Proof. All the properties except the last one are explicit in the axioms for neutral geometry. The proof of the Triangle Inequality splits into two cases.

CASE 1. *The points x, y, z are collinear.* — Let L be a line containing all of these points (we do not necessarily assume they are distinct!), and let $f : L \to \mathbb{R}$ be a 1–1 distance preserving correspondence given by the Ruler Postulate, so that if $u, v \in L$ then $\mathbf{d}(u, v) = |f(u) - f(v)|$. The algebraic inequality $|a + b| \leq |a| + |b|$, which holds for all real numbers a and b, then leads to the following chain of equations and inequalities:

$$\mathbf{d}(x, z) \;=\; |f(x) - f(z)| \;\leq\; |f(x) - f(z)| \;+\; |f(z) - f(y)| \;=\; \mathbf{d}(x, y) \;+\; \mathbf{d}(y, z)$$

Therefore the Triangle Inequality holds if x, y, z are collinear.

CASE 2. *The points x, y, z are* NOT *collinear.* — This follows from the Classical Triangle Inequality (Theorem III.2.6), which is true in neutral geometry by the proof given in `neutral-proofs1.pdf`.■

We can now define open sets, closed sets, continuous functions, Cartesian products and homeomorphisms as in Sutherland's book. The next result indicates some key relationships between such concepts and elementary geometry:

THEOREM 1. *Let* $\mathbf{P}$ *be a neutral plane. Then the following hold:*

(i) *The distance function is a continuous function of two variables.*

(ii) *If* Γ *is a circle, then the interior and exterior regions associated to* Γ *(defined on page* 116 *of the notes) are open subsets of* $\mathbf{P}$*, and* Γ *itself is a closed subset of* $\mathbf{P}$.

(iii) *If* L *is a line in* $\mathbf{P}$ *and* $f : L \to \mathbb{R}$ *is a* 1–1 *distance preserving correspondence given by the Ruler Postulate, then* f *is continuous (and in fact it is a homeomorphism).*

(iv) *If* L *is a line in* $\mathbf{P}$ *and* H *is one of the half-planes it defines, then* H *is open and* L *is closed.*

Proof. The first statement holds for arbitrary metric spaces, and likewise for the second if we define circle, interior regions and exterior regions as on page 116 of the notes. Specifically, part (i) is Exercise 5.17 on page 59 of Sutherland, and (ii) can be derived as follows: Let z be the center of the circle Γ, let r be its radius, and let $g(x) = \mathbf{d}(x, z)$. Then g is continuous, and Γ is the set of points where $g(x) = r$, while the interior and exterior are the sets of points where $g(x) < r$ and $g(x) > r$. Since the first of these subsets defined by the continuous function g is closed and the other two are open, the conclusion follows.

The validity of (iii) follows because a 1–1 onto distance preserving map is always a homeomorphism (see the discussion on pages 71–72 of Sutherland).

Finally, we prove (iv) as follows. Suppose that $p \in \mathbf{P}$ does not lie on L, and let q be the foot of the perpendicular from p to L. If $x \in L$ and $x \neq q$, then we have a right triangle Δpqx (with a right angle at q), and $\mathbf{d}(p, x) \geq \mathbf{d}(p, q)$ because the hypotenuse of a right triangle is longer than either of the other sides (recall the other angles in the triangle are acute, and the longer side is opposite the larger angle). It follows that if $y \in \mathbf{P}$ satisfies $\mathbf{d}(x, y) < \mathbf{d}(p, q)$, then $y \in \mathbf{P} - L$, and consequently $\mathbf{P} - L$ is open in $\mathbf{P}$. It follows that L is closed in $\mathbf{P}$.

To show that each half-plane H is open, let $p \in H$ and choose q as above. It will suffice to show that if $\mathbf{d}(p, y) < \mathbf{d}(p, q)$ then $y \in H$. Assume this is false, and let z be such that $\mathbf{d}(p, z) < \mathbf{d}(p, q)$ but $y \notin H$. Then $z \notin L$, and hence z must lie on the opposite side H' of L. Therefore by the Plane Separation Postulate there is some point w such that $p * w * z$ and $w \in L$. This means that $\mathbf{d}(p, w) < \mathbf{d}(p, z) < \mathbf{d}(p, q)$, so by the preceding paragraph we must have $w \notin L$ — a contradiction. The source of this contradiction is the assumption that there is some point z such that $\mathbf{d}(p, z) < \mathbf{d}(p, q)$ but $z \notin H$, and therefore it follows that $\mathbf{d}(p, y) < \mathbf{d}(p, q)$ implies $y \in H$, and accordingly H is open in $\mathbf{P}$.■

Complement to Theorem 1. *A similar result is valid in a neutral* 3*-space* $\mathbf{S}$ *with "half-space" replacing "half-plane."*

The complement follows if one replaces the application of the Plane Separation Postulate with an application of the Space Separation Postulate.■

We now have the background necessary to prove some standard properties of circles in neutral geometry.

THEOREM III.6.1. (Line-Circle Theorem) *Let L be a line and let Γ be a circle in the neutral plane $\mathbf{P}$, and suppose that L contains a point inside Γ. Then Γ meets L in exactly two points.*

Proof. Let r denote the radius of Γ. It will be convenient to split the proof into two cases.

Suppose first that the line L contains the center of Γ. Then by earlier results we know that L meets Γ in two points.

Suppose now that L does not contain the center z of Γ and let x be a point of L which lies inside Γ. Let u be the foot of the perpendicular from z to L. Since the perpendicular segment from z to L is the shortest segment joining z to a point in L, it follows that $r > \mathbf{d}(z,x) \geq \mathbf{d}(z,u)$. Let $h : \mathbb{R} \to L$ be a distance preserving 1–1 correspondence such that $h(0) = u$, and take $g : \mathbb{R} \to \mathbb{R}$ to be the funtion $g(t) = \mathbf{d}(\,h(t), z)$. Since h is an isometry, it is a homeomorphism and by the joint continuity of the distance function it follows that g is a continuous function. It follows that $g(0) = \mathbf{d}(u,z)$ is the unique minimum of this function, and by the preceding discussion we have $g(0) < r$.

Next, we claim that $g(t) = g(-t)$ for all $t > 0$; to see this, let a and b be the points on L corresponding to $\pm\, t$, so that $\mathbf{d}(u,b) = t = \mathbf{d}(u,a)$. This implies that $\Delta zua \cong \Delta zub$ by SAS, which in turn shows that $g(-t) = \mathbf{d}(z,b) = \mathbf{z}(u,a) = g(t)$. Furthermore, since the Triangle Inequality implies $d(c_1,c_2) \geq d(c_1,c_3) - d(c_2,c_3)$ for all c_1, c_2, c_3, it follows that if $t > 0$ and $h(v) = t$ then

$$g(t) \;=\; \mathbf{d}(v,z) \;\geq\; \mathbf{d}(v,u) \;-\; \mathbf{d}(z,u) \;\geq\; t \;-\; \mathbf{d}(z,u)$$

so that $\lim_{t\to\infty}\; g(t) \;=\; +\infty$. By the Intermediate Value Theorem for continuous functions, there is some $t_0 > 0$ such that $g(t_0) = r$, and by the preceding discussion it follows that $h(\pm\, t_0) \in \Gamma$. Thus the line and circle have at least two points in common.

To conclude the proof, we need to show that the line and circle cannot have three points in common. If this were the case, then we could label three common points a, b, c such that $a * b * c$. By the theorem characterizing perpendicular bisectors, it follows that if k and m are the midpoints of $[ab]$ and $[bc]$ respectively, then $k \neq m$ (in fact, $k * b * m$ is true) and the lines zk and zm are perpendicular to L. This contradicts the uniqueness of perpendiculars to a line through an external point, and the source of this contradiction is our assumption that L and Γ have (at least) three points in common.■

The preceding result implies the following classical theorem on tangents to a circle in neutral geometry.

THEOREM III.6.1A. *Let $\mathbf{P}$ be a neutral plane, let Γ be a circle in $\mathbf{P}$ with center z and radius r, and let $L = xy$ be a line in $\mathbf{P}$ such that $z \notin xy$ and $x \in \Gamma$. Then the following are equivalent:*

(i) $\Gamma \cap xy = \{x\}$.

(ii) *The lines xy and zx are perpendicular.*

Proof. [$(i) \Rightarrow (ii)$] Let M be a perpendicular from z to L, and let w be the point where L and M meet. If $w = x$ then (ii) is true, so assume that $w \neq x$ henceforth. Since the perpendicular segment from z to L is the shortest segment joining z to a point in L, it follows that $r = \mathbf{d}(z,x) >$

$\mathbf{d}(z,w)$. Therefore $w \in L$ lies inside Γ, and thus the Line-Circle Theorem implies that L meets Γ in two points, which contradicts our assumption that $\Gamma \cap xy = \{x\}$. The source of this contradiction was the assumption that $w \neq x$, and therefore we must have $w = x$.■

$[(ii) \Rightarrow (i)]$ Since the perpendicular segment from z to L is the shortest segment joining z to a point in L, it follows that $r = \mathbf{d}(z,x) < \mathbf{d}(z,v)$ for all $v \in L$ such that $v \neq x$. Therefore x is the only point of L which lies on Γ.■

We can also use continuity considerations to give a very short proof of the next result from the notes:

PROPOSITION III.6.2. *Let Γ be a circle in the neutral plane $\mathbf{P}$, and suppose that we have points a and b that are (respectively) inside and outside Γ. Then the open segment (ab) meets Γ in exactly one point.*

Proof. There are two cases depending upon whether the line $L = ab$ passes through the center z of the circle. Let r denote the radius of the circle.

If $z \in L$, let $h : \mathbb{R} \to L$ be a distance preserving 1–1 correspondence such that $h(0) = z$ and $b = h(s)$ where $s > 0$. The hypotheses imply that $s > r$. If we choose s' such that $a = h(s')$, then it follows that $|s'| < r$ so that

$$-r \quad < \quad s' \quad < \quad r \quad < \quad s$$

and if we choose c and c' such that $h(c) = r$ and $h(c') = -r$, then $c \in (ab)$ but $c' \notin (ab)$. Since c and c' are the only two points on $L \cap \Gamma$, then it follows that (ab) meets Γ at c and nowhere else.

Assume now that $z \notin L$. As in previous discussions, let w be the foot of the perpendicular from z to L, let $h : \mathbb{R} \to L$ be a distance preserving 1–1 correspondence such that $h(0) = u$ and $t_b = h^{-1}(b) > 0$, and take $g : \mathbb{R} \to \mathbb{R}$ to be the funtion $g(t) = \mathbf{d}(\,h(t), z)$. As in the proof of Theorem III.6.1, we have $g(t) = g(-t)$ for all t; furthermore, by the hypotheses we also have $g(t_b) > r > g(t_a) \geq g(0)$, where $t_a = h^{-1}(a)$.

Since $a \in L$ lies in the interior of Γ, the Line-Circle Theorem implies that there are two points in $L \cap \Gamma$; if we choose $s' < s$ such that $h(s')$ and $h(s)$ are these common points, then the identity $g(t) = g(-t)$ implies that $s' = -s$ and $s > 0$.

We can now use Theorem III.3.8 and the theorem in `betweenness.pdf` to conclude that a point $p = h(t) \in L$ lies in the interior of Γ if and only if $|t| < s$. Since $t_b > 0$ and $b = h(t_b)$ lies in the exterior of Γ, this means that $t_b > s$. Furthermore, since $a = h(t_a)$ lies in the interior of Γ, we also have $|t_a| < s$. As in the proof for the case where $z \in L$, this implies that $h(s) \in (ab)$ but $h(-s) \notin (ab)$, and therefore (ab) and Γ have exactly one point in common.■

We shall also need the following fundamental consequence of the Line-Circle Theorem.

PROPOSITION III.6.2A. *Let Γ be a circle in a neutral plane $\mathbf{P}$, and let U denote the interior of Γ. Then Γ and $\Gamma \cup U$ are convex sets.*

In Euclidean geometry this can be proved very simply using coordinates, but as usual a synthetic argument is required in neutral geometry.

Proof. If $x, y \in U$ then by the Line-Circle Theorem the line xy meets Γ in two points p and q, and by Theorem III.3.8 we know that $U \cap xy = (pq)$, so that $x, y \in (pq)$. Since (pq) is convex it follows that $(xy) \subset (pq) \subset U$, which shows that U is convex. Similarly, if $x, y \in \Gamma \cup U$ and x or $y \in U$, then $(\Gamma \cup U) \cap xy = [pq]$ by Theorem III.3.8, and hence $[xy] \subset [pq]$ since $[pq]$ is convex. Finally, if $x, y \in \Gamma$ then $(\Gamma \cup U) \cap xy = [xy]$ by Theorem III.3.8. If we combine these observations, we see that $\Gamma \cup U$ is convex.■

Our next goal is to prove the Two Circle Theorem, but before doing we shall prove that two circles can have at most two points in common.

CIRCLE INTERSECTION LEMMA. *If Γ_1 and Γ_2 are distinct circles in the neutral plane* $\mathbf{P}$, *then* $\Gamma_1 \cap \Gamma_2$ *cannot contain three (distinct) points.*

Proof. There are several cases to consider.

CASE 1. Suppose that Γ_1 and Γ_2 have the same center, say z. — Let r_1 and r_2 be the respective radii of the circles. If $x \in \Gamma_1 \cap \Gamma_2$, then we have $r_1 = \mathbf{d}(x,z) = r_2$, which means that $\Gamma_1 = \Gamma_2$. Therefore if the circles have the same center and are distinct, then they are disjoint.∎

For the remainder of the proof, we shall assume that the centers z_1 and z_2 of Γ_1 and Γ_2 are distinct. Let L be the line $z_1 z_2$ joining the centers of the circles, and as in Case 1 let r_1 and r_2 denote their respective radii.

CASE 2. Suppose that Γ_1 and Γ_2 contains at least one point on the line joining the centers. — *In this case, we shall prove that* $\Gamma_1 \cap \Gamma_2$ *consists of a single point.*

Let $f : L \to \mathbb{R}$ be a distance preserving 1–1 correspondence; composing with a self-isometry of $\mathbb{R}$ if necessary, we can find such a map which satisfies the added conditions $f(z_1) = 0 < f(z_2)$, and denote the latter by c. If $p \in \Gamma_1 \cap \Gamma_2$ is a point which lies on L, set $a = f(p)$, and if q is another such point set $b = f(q)$. Then $p, q \in \Gamma_1 \cap \Gamma_2$ translates into the following numerical equations in $\mathbb{R}$:

$$|a| \;=\; r_1 \;=\; |b| \;, \qquad |a-c| \;=\; r_2 \;=\; |b-c|$$

The first of these equations implies that $a = \pm\, b$. We claim that $r_1 > 0$ and $r_2 > 0$ imply that $0 \neq a = b$ in this system of equations. This translates into the geometric statement that $\Gamma_1 \cap \Gamma_2$ can contain at most one point on L.

To prove that $a = b$, note that the squares of the two displayed equations yield a new system

$$a^2 \;=\; b^2 \;, \qquad a^2 - 2ac + c^2 \;=\; b^2 - 2bc + c^2$$

which reduces to the single equation $2ac = 2bc$, and since $c > 0$ this implies $a = b$.

To complete the proof in this case, we need to prove that $\Gamma_1 \cap \Gamma_2$ cannot contain any points which are not on the line L joining the centers of the circles. — The conditions $|a| = r_1 > 0$, $|a - c| = r_2 > 0$ and $c > 0$ imply that $0, a, c$ are distinct real numbers and **exactly one** of the following equations holds, depending upon whether $a < 0 < c$, $0 < a < c$ or $0 < c < a$:

$$|c-a| \;=\; |c-0| \;+\; |a-0| \;, \qquad |c-0| \;=\; |c-a| \;+\; |a-0| \;, \qquad |a-0| \;=\; |a-c| \;+\; |c-0|$$

Since $\mathbf{d}(z_1, z_2) = c > 0$ and the radii r_1, r_2 are given by $|a|$ and $|c - a|$ respectively, the three displayed options translate (respectively) into the following geometrical statements:

$$r_2 \;=\; \mathbf{d}(z_1,z_2) \;+\; r_1 \;, \quad \mathbf{d}(z_1,z_2) \;=\; r_2 \;+\; r_1 \;, \quad r_1 \;=\; \mathbf{d}(z_1,z_2) \;+\; r_2$$

On the other hand, if there is also a point $y \in \Gamma_1 \cap \Gamma_2$ which does not lie on L, so that $\mathbf{d}(x, z_1) = r_1$ and $\mathbf{d}(x, z_2) = r_2$, then the Classical Triangle Inequality implies **all three** of the following inequalities:

$$r_2 \;<\; \mathbf{d}(z_1,z_2) \;+\; r_1 \;, \quad \mathbf{d}(z_1,z_2) \;<\; r_2 \;+\; r_1 \;, \quad r_1 \;<\; \mathbf{d}(z_1,z_2) \;+\; r_2$$

To summarize the conclusions from the two displayed lines, all three inequalities from the second display are valid and exactly one of the equations from the first display is also valid. No matter which one of the equations holds, we have a contradiction. The source of this contradiction is the assumption about the existence of a point $y \in \Gamma_1 \cap \Gamma_2$ which does not lie on L, and consequently no such point can exist.■

CASE 3. Suppose that Γ_1 and Γ_2 contains at least one point, but no points on the line L joining the centers. — *In this case, we shall prove that $\Gamma_1 \cap \Gamma_2$ consists of two points, with one on each side of L.*

Suppose that p is a point in the intersection; we need to find a second point q on the other side of L, and we need to show that there are no other points in the intersection which lie on the same side of L as p (this will also imply that there cannot be two intersection points on the same side of L as q, which is the side of L opposite p).

Intuitively speaking, we get the intersection point on the opposite side of L by taking the mirror image of p with respect L. Formally, if we choose q to be on the side opposite p such that $\mathbf{d}(q, z_2) = r_2 = \mathbf{d}(p, z_2)$ and $|\angle q z_2 z_1| = |\angle p z_2 z_1|$, then by **SAS** we have $|\Delta q z_2 z_1| \cong |\Delta p z_2 z_1|$, which implies that $\mathbf{d}(z_1, q) = \mathbf{d}(z_1, p) = r_1$.

Similar considerations imply that there is only one point in $\Gamma_1 \cap \Gamma_2$ which is on the same side of L as p. If p' is an arbitrary point with these properties, then by **SSS** we have $|\Delta p' z_2 z_1| \cong |\Delta p z_2 z_1|$, which implies that $|\angle p' z_2 z_1| = |\angle p z_2 z_1|$. Since p and p' lie on the same side of L, the Protractor Postulate implies that $[z_2 p = [z_2 p'$, and consequently $p' \in (z_2 p$. Since $\mathbf{d}(p', z_2) = r_2 = \mathbf{d}(p, z_2)$ we must have $p = p'$ by Proposition II.3.1.■

The Two Circle Theorem and one consequence

The Two Circle Theorem is similar in nature to the Line-Circle Theorem and it can also be proved using continuity considerations, but doing so will require considerably more work.

THEOREM III.6.3. (Two Circle or Circle-Circle Theorem) *Suppose that Γ_1 and Γ_2 are two circles in a neutral plane $\mathbf{P}$ with different centers such that Γ_2 contains a point inside Γ_1 and a point outside Γ_1. Then Γ_1 and Γ_2 meet in two points, one on each side of the line joining their centers.*

Observe that the two circles cannot have the same center, for two distinct circles with the same center are disjoint (because the distances of points on the two circles from the common center are unequal).

In the notes, the following result is derived as a consequence of the Two Circle Theorem; exactly the same argument goes through in neutral geometry.

THEOREM III.6.4. (Converse to the Classical Triangle Inequality) *Suppose we are given real numbers $x \geq y \geq z > 0$ which satisfy the condition $x < y + z$. If $\mathbf{P}$ is a neutral plane, then there is a triangle ΔABC in $\mathbf{P}$ such that $\mathbf{d}(B, C) = x$, $\mathbf{d}(A, C) = y$, and $\mathbf{d}(A, B) = z$.*■

To illustrate the preceding result, we shall derive Proposition 1 in Book I of Euclid's *Elements*.

COROLLARY III.6.4A. *Suppose we are given a real number $r > 0$. If $\mathbf{P}$ is a neutral plane, then there is a triangle ΔABC in $\mathbf{P}$ such that $\mathbf{d}(B, C) = \mathbf{d}(A, C) = \mathbf{d}(A, B) = r$.*

Proof. If we take $x = y = z = r$, then $x = r < 2r = y + z$ and consequently x, y and z satisfy the hypotheses in Theorem III.6.4.■

Proof of Theorem III.6.3. Our argument will be based upon the following statement, which is geometrically very plausible but requires a careful proof:

CONTINUITY PRINCIPLE FOR SEMICIRCULAR ARCS. *Let* $\mathbf{P}$ *be a neutral plane, let* $x \neq y$ *be points in* $\mathbf{P}$*, let* z *be a point in* $\mathbf{P} - xy$*, and let* H *be the open half-plane determined by* z *and* xy*. For each real number* $t \in (0, 180)$ *define* $\gamma(t)$ *to be the unique point in* H *such that* $\gamma(t) \in H$*,* $\mathbf{d}(\gamma(t), x) = \mathbf{d}(y, x)$*, and* $|\angle \gamma(t)\, x\, y| = t$*; also, set* $\gamma(0) = y$ *and define* $\gamma(180)$ *to be the point* y' *such that* $y' * x * y$ *and* $\mathbf{d}(x, y') = \mathbf{d}(x, y)$*. Then* $\gamma(t)$ *is continuous on the closed interval* $[0, 180]$.

Simply stated, if Γ is the circle in $\mathbf{P}$ with center x and radius $\mathbf{d}(x, y)$, then the curve $\gamma(t)$ is the semicircular arc of Γ whose endpoints are on the line xy and whose other points lie in H.

Before proving the Continuity Principle, we shall show that it implies the Two Circle Theorem.

Proof of Theorem III.6.3. (*Assuming the Continuity Principle*) Let b_1 and b_2 denote the centers of Γ_1 and Γ_2 respectively, let r_1 and r_2 be the respective radii of these circles, and let $q = \mathbf{d}(b_1, b_2)$. The hypotheses imply that there are points c and c' such that $\mathbf{d}(c, b_2) = r_2 = \mathbf{d}(c', b_2)$ and $\mathbf{d}(b_1, c) < r_1 < \mathbf{d}(b_1, c')$.

We CLAIM that if there are two points which satisfy the hypotheses in the theorem, then it is possible to find two points which satisfy the hypotheses but do NOT lie on opposite sides of the line $b_1 b_2$. — The impact of this assertion is that we might as well assume that c and c' both lie on the union of $L = b_1 b_2$ with a single half-plane and hence lie on the type semicircular arc which appears in the statement of the Continuity Principle.

> Proof of the claim. Suppose that c and c' lie on opposite sides of L (otherwise the original pair of points has the property described in the claim). By the Ruler and Protractor Postulates, there is a point c^* such that c^* and c lie on the same side of L with $\mathbf{d}(c', b_2) = r_2 = \mathbf{d}(c^*, b_2)$ and $|\angle c' b_2 b_1| = |\angle c^* b_2 b_1|$. By **SAS** we have $|\Delta c' b_2 b_1| \cong |\Delta c^* b_2 b_1|$, which implies that $\mathbf{d}(b_1, c') = \mathbf{d}(b_1, c^*)$. Therefore the hypotheses of the theorem hold if we replace the points c and c' with c and c^*.

Let A denote the semicircular arc consisting of all points of Γ_2 which are also on either L or the half-plane H containing c and c', and let γ be the function described in the Continuity Principle whose image is A. Choose s and s' such that $\gamma(s) = c$ and $\gamma(s) = c'$, and define the continuous function $h(t) = \mathbf{d}(\gamma(t), b_1)$. Then we have $h(s) < r_1 < h(s')$, and therefore by the Intermediate Value Theorem there is some θ strictly between s and s' such that $h(\theta) = r_1$; the strict betweenness condition ensures that θ is not an endpoint of $[0, 180]$. If $p = \gamma(\theta)$, then it follows that $p \in \Gamma_2 \cap \Gamma_1$ and p lies on the half-plane containing all of A except the end points, so we have found a common point one of the half-planes defined by L.

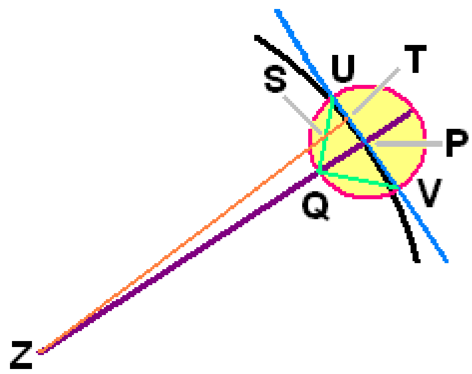
As in the proof of the Circle Intersection Lemma, we get the other intersection point of the circles by taking the mirror image of p with respect to the line L. Formally, if we choose p' to be on the side opposite p such that $\mathbf{d}(p', b_2) = r_2 = \mathbf{d}(p, b_2)$ and $|\angle p' b_2 b_1| = |\angle p b_2 b_1|$, then by **SAS** we have $|\Delta p' b_2 b_1| \cong |\Delta p b_2 b_1|$, which implies that $\mathbf{d}(b_1, p') = \mathbf{d}(b_1, p) = r_1$.

Since the Circle Intersection Lemma implies that the two circles cannot have three points in common, this completes the proof.■

Proof of the Continuity Principle

One major geometric step in the argument is contained in the following result:

LEMMA. *Let* **Z** *and* **P** *be points (in a neutral plane) such that* $d(\mathbf{Z}, \mathbf{P}) = r$. *For each sufficiently small* $\varepsilon > 0$, *there is some* $\delta > 0$ *such that if* $|\angle \mathbf{XPZ}| < \delta$ *and* $d(\mathbf{X}, \mathbf{Z}) = r$, *then* $d(\mathbf{X}, \mathbf{P}) < \varepsilon$.



Our proof will be based on data associated to the drawing above. The points **U** and **V** are chosen such that $\mathbf{U}*\mathbf{P}*\mathbf{V}$, $d(\mathbf{U}, \mathbf{P}) = \varepsilon = d(\mathbf{V}, \mathbf{P})$ and **UV** is perpendicular to **ZP.** We also assume that $\varepsilon < r$ and choose **Q** such that $\mathbf{Z}*\mathbf{Q}*\mathbf{P}$ and $d(\mathbf{P}, \mathbf{Q}) = \varepsilon$ (it follows that $d(\mathbf{Z}, \mathbf{Q}) = r - \varepsilon$); note that **U** and **V** lie on opposite sides of **ZP.** The point **E** will be described more explicitly in the course of the proof; as the drawing suggests, one property will be that **E** lies on the open segment **(QU).** An arc from the circle Γ with center **Z** and radius r is colored in **black**, and the circle with center **P** and radius ε is colored in **red**. By **SAS** we have $\triangle \mathbf{UPZ} \cong \triangle \mathbf{VPZ}$, so that $|\angle \mathbf{UZP}| = |\angle \mathbf{VZP}|$; call this common value α. Note also that if $\mathcal{W}$ is the interior of Γ then Proposition **III.6.2A** implies that $\mathcal{W} \cup \Gamma$ and $\mathcal{W}$ are convex sets.

Let **S** be a point on **(UQ)** such that $d(\mathbf{P}, \mathbf{Q}) < \varepsilon/2$. Then it follows that **S** lies in the interior of $\angle \mathbf{UZP} = \angle \mathbf{UZQ}$. Let $|\angle \mathbf{UZS}| = \delta$; then it follows that $\delta < \alpha$. By the Croosbar Theorem the open ray **(ZS** meets the open segment **(UP)** at some point **T.**

Proof of the Lemma. We claim it will suffice to prove the lemma when **X** lies on the same side of **ZP** as **U;** if this case is known and **X** lies on the same side as **V,** then as before we can find a unique point **Y** on the same side as **U** such that such that $|\angle \mathbf{XZP}| = |\angle \mathbf{YZP}|$ and $d(\mathbf{X}, \mathbf{Z}) = d(\mathbf{Y}, \mathbf{Z}) = r$. By **SAS** we have $\triangle \mathbf{PZX} \cong \triangle \mathbf{PZY}$, so that $d(\mathbf{X}, \mathbf{P}) = d(\mathbf{Y}, \mathbf{P})$. If we already know that the lemma is true for points on the same side of **ZP** as **U,** then we have $d(\mathbf{Y}, \mathbf{P}) < \varepsilon$, and by the preceding sentence this implies the desired inequality $d(\mathbf{X}, \mathbf{P}) < \varepsilon$ for points on the same side of **ZP** as **V.**

In the setting we have developed, we shall prove that *if* **X** *lies on the same side of* **ZP** *as* **U** *with* $|\angle \mathbf{XPZ}| < \delta$ *and* $d(\mathbf{X}, \mathbf{Z}) = r$, *then* $d(\mathbf{X}, \mathbf{P}) < \varepsilon$.

Since X lies on the same side of ZP as U and

$$|\angle XZP| \quad < \quad \delta \quad = \quad |\angle UZP|$$

by Theorem II.3.8 it follows that X lies in the interior of $\angle UZP$. Therefore the Crossbar Theorem implies that $(ZX$ meets (PT) in E and (QS) in F. Since the perpendicular gives the shortest segment from Z to UV, it follows that $\mathbf{d}(Z,E) > r$. Furthermore, since $F \in (QS)$ we have $\mathbf{d}(S,F) < \mathbf{d}(S,Q) < \frac{1}{2}\,\varepsilon$, and hence we also have

$$\mathbf{d}(Z,F) \quad \leq \quad \mathbf{d}(Z,Q) \ + \ \mathbf{d}(Q,S) \quad < \quad (r-\varepsilon) \ + \ \tfrac{1}{2}\varepsilon \quad < \quad r \ .$$

To summarize, the points E and F lie on $(QX$ such that

$$\mathbf{d}(Z,F) \quad < \quad r \quad = \quad \mathbf{d}(Z,X) \quad < \quad \mathbf{d}(Z,E)$$

which implies the betweenness relation $F * X * E$.

We claim that E and F both lie in the interior region $\mathcal{W}$; if so, then $X \in \mathcal{W}$ since the latter is convex, and consequently $\mathbf{d}(P,X) < \varepsilon$. — But by construction $E \in (PT) \subset (UV) \subset \mathcal{W}$ (by Proposition III.3.8), and similarly $F \in (QS) \subset (QU) \subset \mathcal{W}$.■

Proof of the Continuity Principle for semicircular arcs. The definition of the semicircular arc $\gamma(t)$ splits into three cases (namely, $t = 0$, $0 < t < 180$, and $t = 180$), and the proof splits into three cases corresponding to the various definitions. For the sake of convenience, we recall the setting common to all three cases:

> $\mathbf{P}$ is a neutral plane containing the noncollinear points x, y and z, and H is the open half-plane in $\mathbf{P}$ determined by the line $L = xy$ and the point $z \notin L$.

The distance from x to y will be denoted by r.

CASE 1. *Suppose that* $0 < t < 180$. — In this case $\gamma(t)$ is the unique point in H such that $\gamma(t) \in H$, $\mathbf{d}(\gamma(t), x) = \mathbf{d}(y, x)$, and $|\angle \gamma(t)\, x\, y| = t$ (as before, the existence and uniqueness of this point follow from the Ruler and Protractor Postulates. We need to show that for each t and each $\varepsilon > 0$ there is some $\delta > 0$ such that $|s-t| < \delta$ implies $\mathbf{d}\,(\,\gamma(s), \gamma(t)\,) < \varepsilon$; it suffices to prove this for all sufficiently small choices of ε (if δ_0 works for ε_0, then it also works for all ε such that $\varepsilon > \varepsilon_0$), so we shall assume that ε is so small that $\varepsilon < r$ and H contains the open disk of radius ε with center $\gamma(t)$ (this is true for all sufficiently small ε because H is open in $\mathbf{P}$).

We shall now apply the lemma on page 8. Given ε as above, take δ as in that lemma, and suppose that $|s - t| < \delta$. If $s < t$, then by Theorem II.3.8 we have $|\angle\gamma(s)xy| < |\angle\gamma(t)xy|$, so that $|\angle\gamma(t)x\gamma(s)| = t - s < \delta$. On the other hand, if $s > t$, then by Theorem II.3.8 we have $|\angle\gamma(s)xy| < |\angle\gamma(t)xy|$, so that $|\angle\gamma(t)x\gamma(s)| = s - t < \delta$. In either case we can use the lemma to conclude that $\mathbf{d}\,(\,\gamma(t), \gamma(s)\,) < \varepsilon$, which shows that γ is continuous at each t satisfying $0 < t < 180$.

Similar considerations apply if $t = 0$, so that $\gamma(0) = y$; the only differences are (1) we need only choose $\varepsilon < r$, (2) we always have $s > 0$; it follows that γ is continuous at 0. Finally, if $t = 180$, then again it is enough to take $\varepsilon < r$, but a little more work is needed. By definition, $\gamma(180) = y'$ satisfies $y' * x * y$ and $\mathbf{d}(x, y') = \mathbf{d}(x, y)$. If $s \in [0, 180]$ is such that $|180 - s| < \delta$ and $s \neq 180$, then $180 - \delta < s < 180$ and by the Supplement Postulate we have

$$|\angle\gamma(s)xy'| \quad = \quad 180 \ - \ |\angle\gamma(s)xy| \quad = \quad 180 \ - \ s \quad < \quad 180 \ - \ (180-\delta) \quad = \quad \delta \ .$$

Therefore the lemma implies that $\mathbf{d}\,(\,\gamma(t), \gamma(s)\,) < \varepsilon$ when $t = 180$, which shows that γ is continuous at 180.■