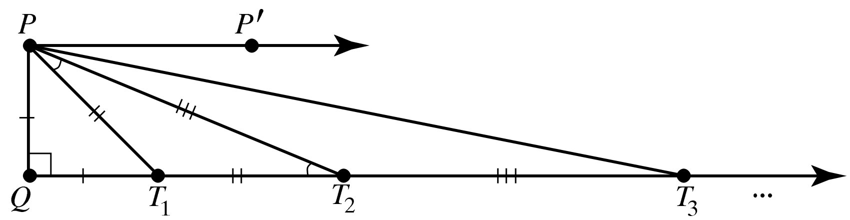
Angle Sum of Triangles in Hyperbolic Geometry

We will now show that the sum of the angle measures in a triangle on the hyperbolic plane is strictly less than 180°. First we need

Lemma. *Suppose $\overline{PQ}$ is a segment and Q' is a point such that $\angle PQQ'$ is a right angle. For every $\epsilon > 0$ there exists a point T on $\overrightarrow{QQ'}$ such that $m(\angle PTQ) < \epsilon°$.*

Proof.



Construct a point P' such that $\overleftrightarrow{PP'}$ is perpendicular to $\overleftrightarrow{PQ}$. We choose a sequence of points $T_1, T_2, \ldots$ as follows. First choose T_1 such that $\overline{PQ} \cong \overline{QT_1}$. Then choose T_2 such that $\overline{PT_1} \cong \overline{T_1T_2}$. Iteratively choose T_n beyond T_{n-1} such that $\overline{PT_{n-1}} \cong \overline{T_{n-1}T_n}$.

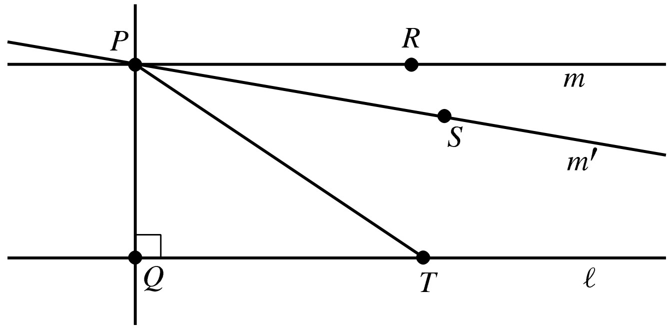
1. Show that $\angle QT_nP \cong \angle T_{n-1}PT_n$.

2. Show that for each n, $m(\angle QPT_1) + m(\angle T_1PT_2) + \ldots + m(\angle T_{n-1}PT_n) < 90°$.

3. Finish the proof using contradiction.

Theorem. *On the hyperbolic plane, there is a triangle whose angle sum is strictly less than* $180°$.

Proof.



Let ℓ be a line and P a point off ℓ. Drop a perpendicular line from P to ℓ, intersecting at Q. Let m be the line through P that is perpendicular to $\overleftrightarrow{PQ}$.

1. Why is m is parallel to ℓ?

By the Hyperbolic Parallel Postulate (HPP), there is another line m' through P that is parallel to ℓ. Choose point S on m' to be on the same side of m as Q, and point R on m to be on the same side of $\overleftrightarrow{PQ}$ as S. Using the Lemma, choose T on ℓ so that $m(\angle QTP) < m(\angle SPR)$.

2. Show that $S(\triangle QTP) < 180°$.

Source:
http://www.math.unl.edu/~bharbourne1/M812TSummer2014/Day8/HyperbolicTriangularAngleSum.pdf