RECTANGLES IN NEUTRAL GEOMETRY

This document provides details for the proof of Theorem **V.3.8** in the course notes**:**

Suppose that a neutral plane $\mathcal{P}$ contains at least one rectangle. Then for each pair of positive real numbers p and q there is a rectangle **ABCD** *such that the lengths of* **[AB]** *and* **[CD]** *are equal to p and the lengths of* **[AD]** *and* **[BC]** *are equal to q.*

There are two significant differences in notation from the course notes**:** First of all**,** the distance between two points **X** and **Y** is denoted by **|XY|** in this document**.** Also**,** the geometric relationship "**Y** is between **X** and **Z**" is denoted by **X — Y — Z.**

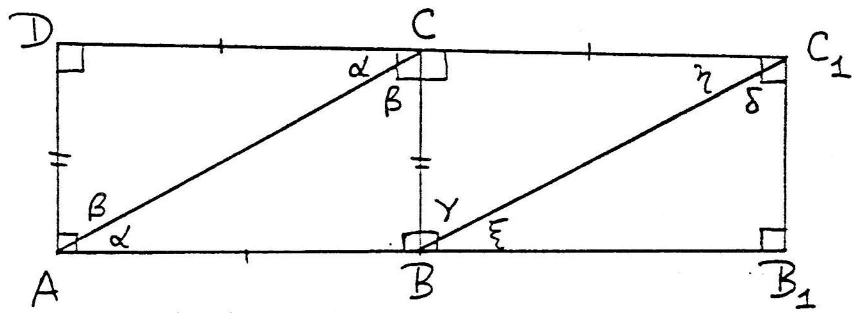
Throughout this document**,** ***all points are assumed to lie in some fixed neutral plane*** $\mathcal{P}$**.**

Note. It may well be that NO rectangles exist in a specific neutral plane. However, the next result says that if one rectangle exists then rectangles with arbitrary lengths and widths also exist.

THEOREM 3. If one rectangle exists, then for each $r, s > 0$ there is a rectangle ABCD with $|AB| = |CD| = r$ and $|BC| = |AD| = s$.

The proof is rather long, and several major steps in the argument will be presented as lemmas.

LEMMA 4 (*Splicing Property*). Suppose that ABCD is a rectangle, and let $C_1 \in [DC$ be a point such that $|DC_1| = 2|DC|$. Let B_1 be the foot of the perpendicular from C_1 to AB. Then $|\angle DC_1B| = 90°$ and the points A, B_1, C_1, D are the vertices of a rectangle.



PROOF. First of all, the lines AD, BC, and B_1C_1 are all parallel to each other because every two of them have a common perpendicular (namely AB). Therefore AD and B_1C_1 are contained in the D - and C_1 - sides of BC respectively. But $|DC_1| = 2|DC|$ and $C_1 \in (DC$ imply $D - C - C_1$ is true. This in turn implies that D and C_1 are on opposite sides of BC. Since B is the common point of the lines AB_1 and BC, it follows that $A - B - B_1$ is true.

Since AD and B_1C_1 are parallel (they have a common perpendicular), the points B_1 and C_1 lie on the same side of AD. Hence A, B_1, C_1, D form the vertices of a convex quadrilateral. Likewise B, B_1, C_1, C form the vertices of a convex quadrilateral.

By construction, S.A.S. applies to show $\Delta ADC \cong \Delta BCC_1$. It follows that $|AC| = |BC_1|$, $\gamma = |\angle CBC_1| = |\angle DAC| = \alpha$, and $\eta = |\angle BC_1C| = |\angle ACD|$. On the other hand, if $\xi = |\angle C_1BB_1|$ then $\alpha + \beta = 90° = \gamma + \xi$. Then $\alpha = \gamma$ implies $\beta = \xi$.

By A.A.S. it follows that $\Delta ABC \cong \Delta BB_1C_1$, and hence $\alpha = \delta = |\angle BC_1B_1|$. This implies that $\eta + \delta = 90°$. But then it follows that $|\angle DC_1B_1| = \eta + \delta = 90°$ ∎

LEMMA 5. <u>If there is a rectangle</u> $ABCD$ <u>in the neutral plane under consideration, then for all</u> $n > 0$ <u>there is a rectangle</u> $A'B'C'D'$ <u>with</u>

$$|A'B'| = |C'D'| = n|AB| = n|AC|,$$
$$|B'C'| = |A'D'| = |BC| = |AD|.$$

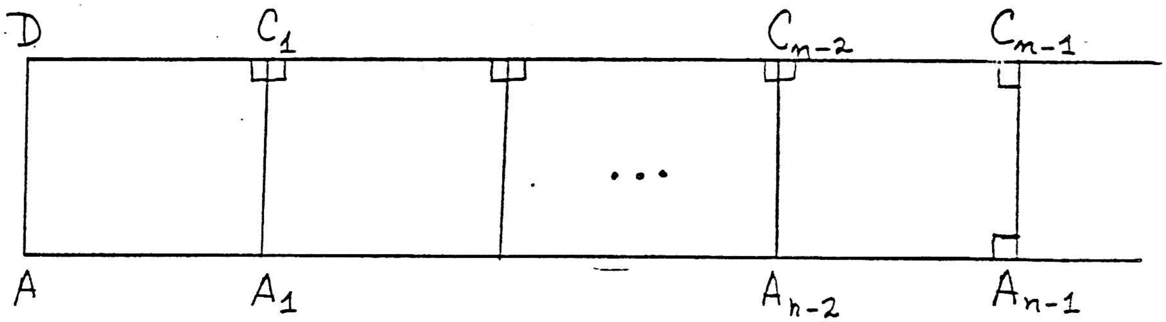
PROOF. The case $n = 2$ was done in the preceding lemma. Assume by induction that we have

$$B = A_1, A_2, \ldots, A_n - 1$$
$$C = C_1, C_2, \ldots, C_{n-1}$$

such that

$$C_0 = D - C_1 - \ldots - C_{n-1},$$
$$A_0 = A - A_1 - \ldots - A_{n-1},$$

$|AB| + |CD| = |A_{i+1}A_i| = |C_{i+1}C_i|$, and $C_iA_i \perp AB, CD$ (perpendicular to both).



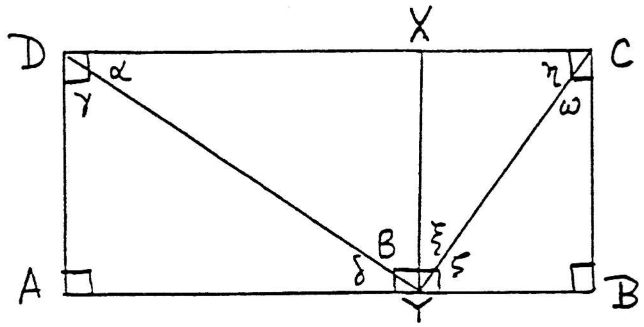
Apply Lemma 4 to the rectangle $A_{n-2}\ A_{n-1}\ C_{n-1}\ C_{n-2}$ to get A_n and C_n with $A_{n-2} - A_{n-1} - A_n$, $C_{n-2} - C_{n-1} - C_n$, $|A_n\ A_{n-1}| = |C_n\ C_{n-1}| = |AB| = |CD|$, and $A_nC_n \perp AB, CD$ ■

COROLLARY 6. *If a rectangle* ABCD *exists, then for arbitrary positive integers* m *and* n *there is a rectangle* A'B'C'D' *with* $|A'B'| = n|AB|$, $|B'C'| = m|BC|$.

PROOF. Apply the preceding lemma to get a rectangle A"B"C"D" with $|A''B''| = n|AB|$, $|B''C''| = |BC|$. Now apply the lemma once again to get a new rectangle A'B'C'D' with $|A'B'| = |AB|$, $|B'C'| = m|B''C''|$. It follows that $|A'B'| = n|AB|$ and $|B'C'| = m|BC|$ ■

The next result allows us to take a large rectangle and trim it to a smaller size.

LEMMA 7. *Let* ABCD *be a rectangle, let* $X \in (CD)$, *and let* Y *be the foot of the perpendicular from* X *to* AB. *Then* $Y \in (AB)$ *and* A, Y, X, D *are the vertices of a rectangle.*



PROOF. The lines AD, XY, and BC are all parallel (they are all perpendicular to AB). Hence AD $\subseteq$ D-side XY and BC $\subseteq$ C-side XY. But C-X-D (since X lies on (BC)) implies that C and D lie on opposite sides of XY. Hence AD and BC also lie entirely on opposite sides of XY. It follows that (AB) $\cap$ XY $\neq \emptyset$. Since AB $\cap$ XY = {X}, this implies A-X-B must be true.

Label the angle measures as indicated in the diagram above:

$$\alpha = |\angle YDX| \qquad \xi = |\angle CYX|$$
$$\beta = |\angle DYX| \qquad \eta = |\angle XCY|$$
$$\gamma = |\angle ADY| \qquad \zeta = |\angle CYB|$$
$$\delta = |\angle AYD| \qquad \omega = |\angle YCB|.$$

Since AD$\|$XY, BC$\|$XY, and AB$\|$CD, it follows that A,Y,X,D and Y,B,C,X form the vertices of a convex quadrilateral. Therefore we have D $\in$ Int $\angle$AYX, Y $\in$ Int $\angle$ADX, C $\in$ Int $\angle$XYB, and Y $\in$ Int $\angle$XCB. These imply the following four equations:

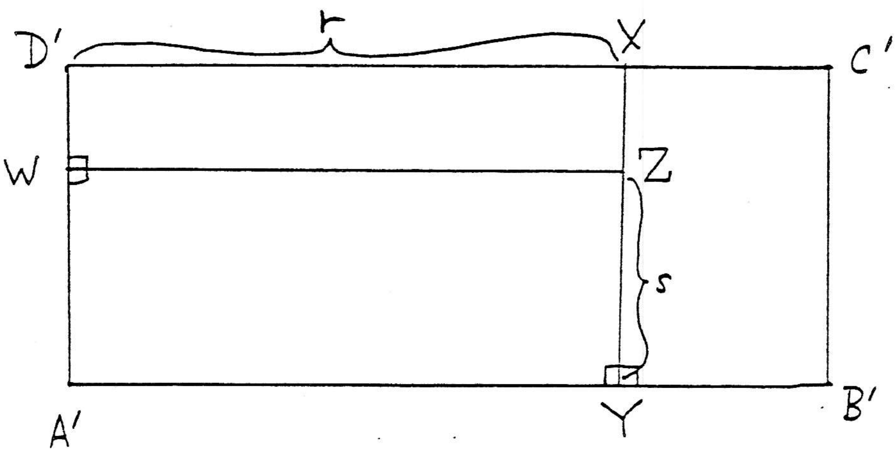
$$\alpha + \gamma = 90° \qquad \xi + \zeta = 90°$$
$$\delta + \beta = 90° \qquad \eta + \omega = 90°.$$

The Saccheri-Legendre Theorem implies the following additional inequalities:

$$\gamma + \delta \leq 90° \qquad \zeta + \omega \leq 90°.$$

These together imply $\alpha + \beta \geq 90°$ and $\xi + \eta \geq 90°$. Therefore the Saccheri-Legendre Theorem implies $|\angle DXY|, |\angle CXY| \leq 90°$. On the other hand, C-X-D implies $180° = |\angle DXY| + |\angle CXY|$. This can happen only if both $|\angle DXY|$ and $|\angle CXY|$ are equal to 90°. But this now implies XY is perpendicular to CD, so that A,Y,X,D form the vertices of a rectangle ∎

PROOF OF THEOREM 3. Given rectangle ABCD and $r, s > 0$, first find positive integers n and m so that $r < n|AB|$ and $s < m|AD|$. Form A'B'C'D' with $|A'B'| = n|AB|$, $|C'D'| = m|CD|$. Let



$X \in (D'C')$ so that $|D'X'| = r$, and let Y be the foot of the perpendicular from X to A'B'. Then by Lemma 3.7 one obtains rectangle A'YXD' with $|A'Y| = r$ and $|YX| = |A'D'| = m|AD|$. Now let $Z \in (YX)$ satisfy $|ZY| = s$, and let W be the foot of the perpendicular from Z to A'D'. Then A',Y,Z,W form the vertices of a rectangle with $|A'Y| = r$ and $|YZ| = s$ ∎