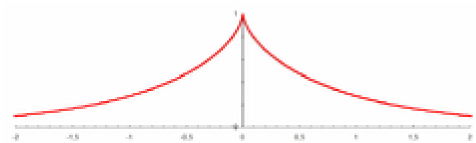
HYPERBOLIC GEOMETRY AND THE PSEUDOSPHERE

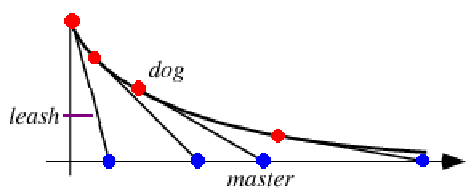
Here is some more detailed information about how the pseudosphere represents a portion of the hyperbolic plane in the Poincaré disk model. Recall that the ***pseudosphere*** is the surface of revolution about the $x-$ axis defined by the ***tractrix***, which is given parametrically by the following equations:

$$y(t) \;=\; 1/\cosh t\,, \qquad\qquad x(t) \;=\; t - \tanh t$$



http://upload.wikimedia.org/wikipedia/commons/thumb/e/ed/TractrizFig1.png/240px-TractrizFig1.png

A standard physical interpretation of the tractrix is depicted below. We start with a chain or rope of fixed length, with the person pulling the chain starting out at the origin and the object being pulled starting out at a point on the $y-$ axis in the upper half $-$ plane. It is assumed that the chain is pulled tight at all times and that the object being pulled does not move on its own (one can question whether these assumptions are realistic for pulling a dog with a leash or a child dragging a pull toy, but in some cases these are reasonable idealizations).



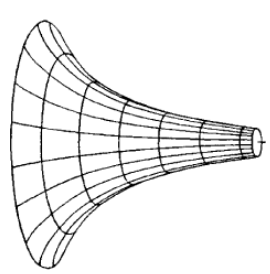
http://mathworld.wolfram.com/images/eps-gif/TractrixDiagram_800.gif

See **http://en.wikipedia.org/wiki/Tractrix** for an animated video.

Parametric equations for the pseudosphere are then given by a standard surface of revolution formula

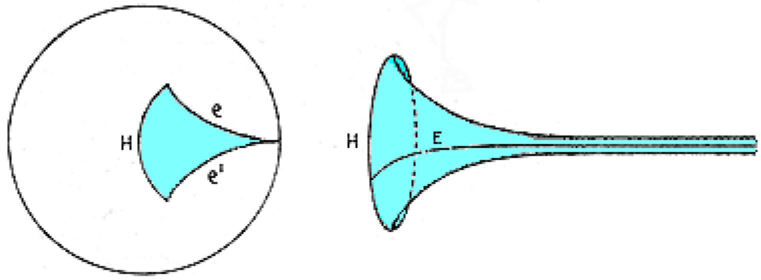
$$\sigma(t,\,\theta) \;=\; (x(t),\; y(t)\cos\theta,\; y(t)\sin\theta)$$

where θ ranges from (say) 0 to 2π and t ranges over all nonnegative integers. Here is an illustration:



(***Source:*** **http://img.tfd.com/ggse/19/gsed_0001_0014_0_img3538.png**)

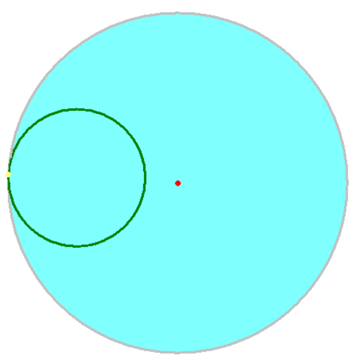
Suppose now that we cut the pseudosphere along its intersection **E** with the $x-$ axis and only take the portion of the pseudosphere where $x > 1$ and t is positive. This cut open piece of the pseudosphere will then correspond isometrically to the light blue region of the Poincaré plane in the drawing below (the pseudosphere continues to infinity at the right of the picture).



(***Source:*** **http://www.cs.unm.edu/~joel/NonEuclid/horn.gif**)

Euclidean surfaces in hyperbolic space

There is also an important family of surfaces in hyperbolic $3-$ space which have the same geometrical structure as the Euclidean plane. A surface in this family is called a ***horosphere***, and it appears that such surfaces were first mentioned in writings of F. L. Wachter (1792 – 1817), who was a student of Gauss. If we intersect a horosphere **H** with a (hyperbolic) plane **P** such that **H** and **P** are perpendicular wherever they meet, then the resulting curve is called a ***horocycle***. In the drawing of the Poincaré disk model given below, the dark green circle, which is tangent to the boundary of the Poincaré disk at the yellow point, is a typical horocycle. If we rotate everything about the $x-$ axis, then we obtain the Poincaré disk model for hyperbolic $3-$ space, and the surface of revolution determined by the dark green circle (with the point of tangency deleted) will be a typical horosphere in this model.



(***Source:*** **http://jwilson.coe.uga.edu/EMAT6680Fa05/Evans/Assignment%207/Tangent%20Circles_files/image004.gif**)

The study of horospheres and horocycles was fundamentally important to the developments of hyperbolic geometry in the work of Bolyai and Lobachevsky. Here are two references for additional information:

http://en.wikipedia.org/wiki/Horocycle

http://en.wikipedia.org/wiki/Horosphere