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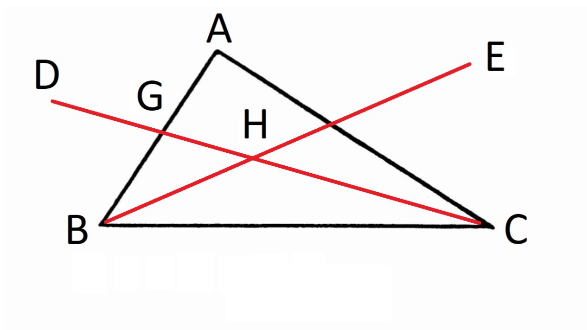
Mathematics 133, Fall 2021, Examination 1

Answer Key

1. [*25 points*] Suppose we are given $\triangle ABC$ and points $D \in \text{Int}\,\angle ACB$ and $E \in \text{Int}\,\angle ABC$. Prove that the open rays $(BE$ and $(CD$ have a point in common

SOLUTION

By the Crossbar Theorem we know that $(CD$ and (AB) have some point G in common. Note that $\angle ABC = \angle GBC$ as sets, so that $E \in \text{Int}\,\angle GBC$.

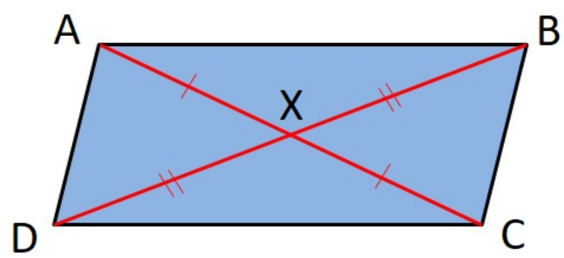


We can now apply the Crossbar Theorem to conclude that there is a point $H \in (BE \cap (CG)$. Since $(CG) \subset (CD$ we also have $H \in (BE \cap (CD$.■

2. [*25 points*] Let $[AC]$ and $[BD]$ be noncollinear closed segments which have a single point X in common, and assume this point is the midpoint of both $[AC]$ and $[BD]$. Prove that AB and CD are parallel lines without using the Euclidean Parallel Postulate (or a logically equivalent statement).

SOLUTION

Here is a drawing:



The common midpoint X satisfies the betweenness conditions $A * X * C$ and $B * X * D$. If we combine this with $|AX| = |CX|$ and $|BX| = |DX|$ and the Vertical Angle Theorem, we find that $\triangle AXB \cong \triangle CXD$ by SAS. It follows that $|\angle XAB| = |\angle XCD|$, and since $\angle XAB = \angle CAB$ and $\angle XCD = \angle ACD$ as sets, it follows that $|\angle CAB| = |\angle ACD|$.

By construction AC is a transversal for the lines AB and CD, and the betweenness conditions imply that B and D lie on opposite sides of BD. Therefore $\angle CAB$ and $\angle ACD|$ are alternate interior angles, and since their measures are equal these lines must be parallel. — Note that this proof does not require the Euclidean Parallel Postulate.■

3. [*25 points*] Assume the plane under consideration is Euclidean, and suppose that we are given $\triangle ABC$ such that $|BC| \leq |AC| \leq |AB|$. Prove that $|\angle ACB| \geq 60°$. [*Hint:* If x, y, z are positive real numbers, why is at least one of them greater than or equal to $\frac{1}{3}(x + y + z)$?]

SOLUTION

We shall first answer the question in the hint. If we have $0 < x, y, z < \frac{1}{3}(x+y+z)$ and we add the associated inequalities for each of x, y, z we obtain the contradiction $x+y+z < 3 \cdot \frac{1}{3}(x + y + z) = x + y + z$. The source of the contradiction is the assumption that $0 < x, y, z < \frac{1}{3}(x + y + z)$, so this must be false and hence at least one x, y, z must be greater than or equal to $\frac{1}{3}(x + y + z)$.

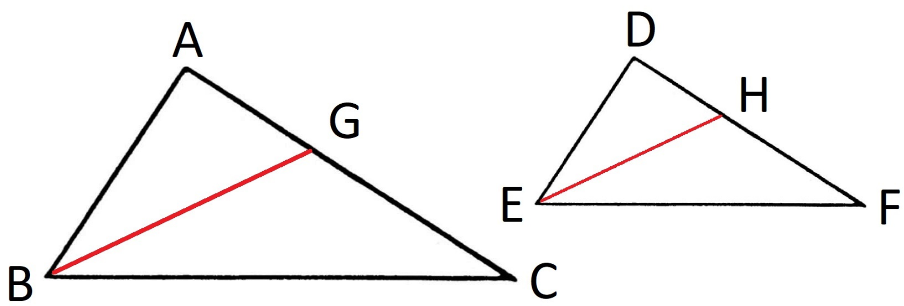
Since the larger angle is opposite the longer side, it follows that $|\angle BAC| \leq |\angle ABC| \leq |\angle ACB|$. Since the sum of these numbers is $180°$, the observation in the preceding paragraph implies that the largest of them, namely $|\angle ACB|$, must be at least $60°$. ∎

4. [25 *points*] Assume the plane under consideration is Euclidean, and suppose that we are given $\triangle ABC \sim \triangle DEF$. If $G \in (AC)$ and $H \in (DF)$ are such that $[BG$ and $[EH$ bisect $\angle ABC$ and $\angle DEF$ respectively, prove that

$$\frac{|BG|}{|BC|} = \frac{|EH|}{|EF|} .$$

SOLUTION

Here is a drawing:



By the similarity assumption we have $|\angle ABC| = |\angle DEF|$ and $|\angle ACB| = |\angle DFE|$. Therefore the bisector conditions imply that $|\angle GBC| = \frac{1}{2}|\angle ABC| = \frac{1}{2}|\angle DEF| = |\angle HEF|$. Therefore the AA Similarity Theorem implies that $\triangle GBC \sim \triangle HEF$, so that

$$\frac{|BG|}{|EH|} = \frac{|BC|}{|EF|} .$$

This is equivalent to the desired equation

$$\frac{|BG|}{|BC|} = \frac{|EH|}{|EF|}$$

because the each of the proportionality equations $p/q = r/s$ and $p/r = q/s$ implies the other.■