1. [*25 points*] Suppose that we are given a convex quadrilateral $ABCD$ in a neutral plane such that $|\angle DAB| = 90^\circ = |\angle BCD|$ and $d(A,B) = d(C,D)$.

(i) Prove that $|\angle ABC| = |\angle CDA|$ and $d(B,C) = d(A,D)$. [*Hint:* First split the quadrilateral into two triangles along diagonal $[BD]$, then do the same thing along diagonal $[AC]$.]

(ii) Explain why the quadrilateral is a rectangle if and only if the plane is Euclidean.

SOLUTION

There are drawings for this exercise on the next page.

(i) By **HS** for right triangles we have $\Delta DAB \cong \Delta BCD$. Therefore $d(B,C) = d(A,D)$. Next by **SSS** we have $\Delta ABC \cong \Delta CDA$. Therefore $|\angle ABC| = |\angle CDA|$.■

(ii) The angle sum of the convex quadrilateral is equal to

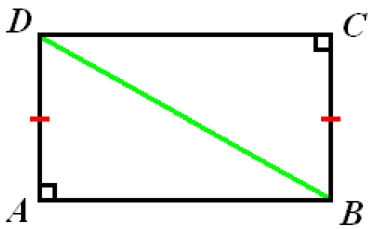
$$180^\circ \ + \ 2 \cdot |\angle ABC| \ = \ 180^\circ \ + \ |\angle ABC| \ + \ |\angle CDA| \ = \ 180^\circ \ + \ 2 \cdot |\angle CDA|$$

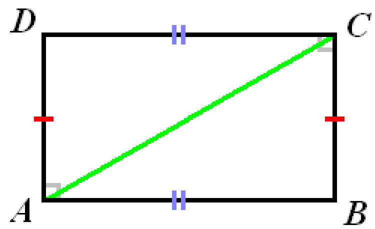
and by a corollary to the Saccheri-Legendre Theorem this sum is $\leq 360^\circ$. If the convex quadrilateral is a rectangle, then it follows that the plane is Euclidean. Conversely, if the plane is Euclidean, then the angle sum of the convex quadrilateral is equal to 360°, and by the formulas in the first sentence this implies that $|\angle ABC| = |\angle CDA| = 90^\circ$, so that the convex quadrilateral is a rectangle.■

Comment. The use of the term "split" in the hint should not have been interpreted as an assumption that the diagonals $[AC]$ and $[BD]$ in the given quadrilateral bisect the angles whose vertices are at their endpoints (A or C in the first case, b and D in the second). In particular, this is never the case in Euclidean geometry for rectangles which are not squares.

Drawings to accompany the solution to Problem 1

Both drawings are for part (i) of the problem.





The first step is to show that $\triangle DAB \cong \triangle BCD$, and this has to be done using the **HS** (hypotenuse – side) congruence theorem for right triangles, ***which by the results in the course notes is valid in neutral geometry***. One consequence of this congruence is that $d(A, B) = d(C, D)$, and the latter yields the data in the drawing on the right, and an application of **SSS** then implies that $\triangle ABC \cong \triangle ADC$. Of course, the latter in turn shows that $|\angle ABC| = |\angle ADC|$.■

2. [*20 points*] Suppose we are given a hyperbolic plane **P** and a small real number $h > 0$. Prove that there is a triangle ΔABC in **P** whose angle defect $\delta(\Delta ABC)$ is less than h. [*Hint:* If we are given ΔDEF and $G \in (EF)$, why is at least one of $\{\delta(\Delta DEG), \delta(\Delta DGF)\}$ less than or equal to $\frac{1}{2}\,\delta(\Delta DEF)$?]

SOLUTION

We know that $\delta(\Delta DEF) = \{\delta(\Delta DEG) + \delta(\Delta DGF)\}$. Given three positive real numbers a, b, c such that $a + b = c$, we have either $a \leq \frac{1}{2}c$ or $b \leq \frac{1}{2}c$ for otherwise we would have $a > \frac{1}{2}c$ and $b > \frac{1}{2}c$, which would imply that $a + b > c$. Combining these, we see that the assertion in the hint — namely, at least of $\{\delta(\Delta DEG), \delta(\Delta DGF)\}$ is less than or equal to $\frac{1}{2}\,\delta(\Delta DEF)$ — must be true.

We can reformulate the preceding to state that given ΔDEF there is some triangle $\Delta D_1E_1F_1$ such that $\delta(\Delta D_1E_1F_1) \leq \frac{1}{2}\delta(\Delta DEF)$. Repeating this argument n times for an arbitrary positive integer n we obtain a triangle $\Delta D_nE_nF_n$ such that $\delta(\Delta D_nE_nF_n) \leq \left(\frac{1}{2}\right)^n \delta(\Delta DEF)$. If $h > 0$ then we know that there is some value of n such that the right hand side is less than h, and for this choice of n we have $\Delta D_nE_nF_n$ such that $\delta(\Delta D_nE_nF_n) < h$.■

3. [*15 points*] Assume that everything in this exercise lies in some Euclidean plane.

(i) Define the orthocenter of ΔABC.

(ii) State the Two Circle Theorem.

SOLUTION

(i) This is the point where the altitudes (perpendiculars from A to BC, B to AC and C to AB) meet.■

(ii) As indicated in the hint written on the board, this is a major result from Section III.6 of the course notes:

Let Γ_1 and Γ_2 be two circles with centers Q_1 and Q_2 respectively. If Γ_2 contains a point in the interior of Γ_1 and a point in the exterior of Γ_1, then $\Gamma_1 \cap \Gamma_2$ consists of exactly two points, with one on each side of the line Q_1Q_2 joining their centers.■

4. [*20 points*] Suppose that we are given four lines L_1, L_2, M_1, M_2 in a Euclidean plane such that $L_1 \perp M_1$, $L_2 \perp M_2$, and L_1 meets L_2 at some point X. Prove that the lines M_1 and M_2 have a point in common. You may use the following theorems: (1) *If M and N are parallel lines and $K \perp M$, then $K \perp N$.* (2) *Two lines perpendicular to a third line are parallel.*

SOLUTION

Suppose that the conclusion is false, so that $M_1 \,||\, M_2$. By $L_1 \perp M_1$ and the first theorem stated in the problem, this means that $L_1 \perp M_2$. If we combine this with $L_2 \perp M_2$ and the second theorem, we conclude that $L_1 \,||\, L_2$. But this contradicts our hypothesis that $L_1 \perp L_2$, and thus our assumption that $M_1 \cap M_2 = \emptyset$ must be false, which means that M_1 and M_2 have a point in common.■

5. [*25 points*] Suppose that we are given $\Delta ABC \sim \Delta AEF$ in the standard Euclidean coordinate plane, where $E \in (AB)$ and $F \in (AC)$.

(i) Vector formulas for E and F are given by $E = A + s(B - A)$ and $F = A + t(C - A)$ where $0 < s, t < 1$. Explain why $s = t$.

(ii) Prove that the lines EF and BC are parallel. You may use the fact that neither E nor F lies on BC.

SOLUTION

(i) Let k be the ratio of similitude. Then we have

$$k = \frac{|E - A|}{|B - A|} = \frac{s|B - A|}{|B - A|} = s$$

$$k = \frac{|F - A|}{|C - A|} = \frac{t|C - A|}{|C - A|} = t$$

and if we combine these equations we see that $s = k = t$.■

(ii) By the conclusion to the first part we have

$$E - F = (E - A) - (F - A) \;\; k(C - A) - k(B - A) = k(C - B)$$

where $k \neq 0$ is the ratio of similitude, and since $E - F$ is a nonzero multiple of $C - B$ the lines EF and BC must be parallel.■

6. [*20 points*] (i) Suppose we are given $\angle BAC$ and a point $D \in (BC)$. Explain why D lies in the interior of $\angle BAC$.

(ii) Let $A \neq B$ be points, and let $f : AB \to \mathbf{R}$ be a 1–1 correspondence such that $d(X, Y) = |f(X) - f(Y)|$ for all points X, Y on the line AB and $f(A) > f(B)$. If C is a third point on AB, state the inequality or inequalities corresponding to the (separate) statements $A * C * B$ and $A * B * C$.

SOLUTION

(i) The ordering relation $B * D * C$ and basic theorems on betweenness and separation imply that (a) B and D lie on the same side of AC, (b) C and D lie on the same side of AB. Since the interior of $\angle BAC$ is the set of all points on the same side of AB as C and also on the same side of AC as B, this means that D lies in the interior of $\angle ABC$.■

(ii) $A * C * B$ corresponds to the inequality chain $f(A) > f(C) > f(B)$, and $A * B * C$ corresponds to the inequality $f(B) > f(C)$.■

7. [*25 points*] In a Euclidean plane, a representative pair of noncongruent triangles satisfying **SSA** are given by ΔABC and ΔABD where $B * C * D$ and $d(A,C) = d(A,D)$. Determine the value of the following expression involving the angles with unequal measures:

$$|\angle DAB| - |\angle CAB| + 2\,|\angle ADB|$$

SOLUTION

To simplify the algebra we shall write the various angle measures as follows:

$$|\angle ABC = \angle ABD| \;=\; \beta \;, |\angle ADC = \angle ADB| \;=\; \delta$$
$$|\angle BAC| \;=\; \alpha_1 \;, |\angle CAD| \;=\; \alpha_2$$
$$|\angle ACB| \;=\; \gamma_1 \;, |\angle ACD| \;=\; \gamma_2$$

There is a drawing on the next page.

The betweenness relation $B{*}C{*}D$ implies that C lies in the interior of $\angle BAD$, and therefore by the Addition Postulate for angle measures we have $|\angle BAD| = \alpha_1 + \alpha_2$. Furthermore, the Supplement Postulate implies that $180° = \gamma_1 + \gamma_2$. Finally, the Isosceles Triangle Theorem implies that $\gamma_2 = \delta$.

Since the angle sum of a Euclidean triangle is 180°, we also have

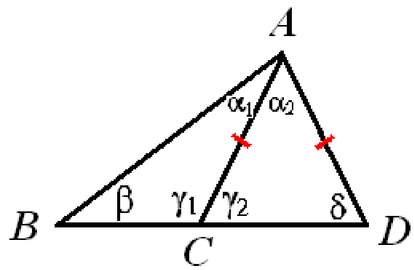
$$\alpha_1 + \alpha_2 + \beta + \delta \;=\; 180° \;, \quad \alpha_1 + \beta + \gamma_1 \;=\; 180° \;, \quad \alpha_2 + \delta + \gamma_2 \;=\; 180°$$

and therefore the expression in the problem is equal to

$$(\alpha_1 + \alpha_2) \;-\; \alpha_1 \;+\; 2\delta \;=\; \alpha_2 + 2\delta \;=\; \alpha_2 \;+\; \delta \;+\; \gamma_2 \;=\; 180°$$

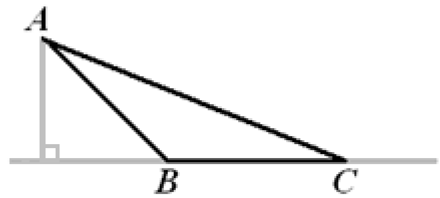
so the expression in the problem is equal to 180°.■

Drawing to accompany the solution to Problem 7



Note on triangles satisfying the SSA criterion

As the wording of Problem **7** suggest, if $\triangle ABC$ and $\triangle XYZ$ (with the associated vertex orderings) satisfy the **SSA** criteria $d(A,B) = d(X,Y)$, $d(A,C) = d(X,Z)$, $|\angle ABC| = |\angle XYZ|$, and in addition $|\angle XYZ|$ is **NOT** a right angle (so we avoid issues involving the **HS** congruence theorem for right triangles), then either $\triangle ABC \cong \triangle XYZ$ or $\triangle ABD \cong \triangle XYZ$. One can view this as part of a standard problem in trigonometry; namely, given real numbers b and c along with an angle measure β, determine the remaining measurements of all triangles $\triangle ABC$ such that $d(A,B) = c$, $d(A,C) = b$, and $|\angle ABC| = \beta$. If $\angle ABC$ is an acute angle, then there are **0**, **1** or **2** possibilities for the remaining measurements depending upon whether b is less than, equal to, or greater than $c \sin \beta$, and in this case the setting of the exercise presents both possibilities for the third case. Several of the online references below provide further information about these cases. On the other hand, if $\angle ABC$ is either a right or obtuse angle, then there is at most one possibility for the remaining measurements, and such a triangle exists if and only if b is greater than c (see the drawing below).



One way of seeing the uniqueness of such triangles is to use Corollary **III.3.2** in the notes. If one could find a second point D on $(BC$ such that, say, $B{*}C{*}D$ and $d(A,D) = d(A,C)$, then $\angle ADB$ and $\angle ACB$ would be an acute angles by that result, and by the Isosceles Triangle Theorem the same would hold for $\angle ACD$. But this is impossible because $\angle ACD$ and $\angle ACB$ are supplementary.■

Finally, here are some online references concerning noncongruent triangles satisfying **SSA:**

http://www.regentsprep.org/Regents/math/algtrig/ATT12/lawofsinesAmbiguous.htm

http://www.ehow.com/how_8680797_solve-triangles-ambiguous-case.html

http://teachers.henrico.k12.va.us/math/ito_08/10AdditionalTrig/10les1/ambiguous_act.pdf

http://mathforum.org/mathimages/index.php/Ambiguous_Case

http://www.algebra.com/algebra/homework/Trigonometry-basics/change-this-name8950.lesson