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Mathematics 133, Fall 2020, Examination 1

Answer Key

1. [*25 points*] Assume that we are given a line L in the plane P, and also assume that A, B, C, D, E are five distinct points such that no three are collinear. Prove that at least two of these points lie on the same side of L in P. [*Note:* One or more points might lie on L.]

SOLUTION

Start with the hint. One or even two points may lie on L, but no more can do so because no three points are collinear. Therefore there is a set with 3 to 5 points which do not lie on L. The complement of L consists of two open half-planes H_+ and H_-, and since we have a set with at least three points, at least two of them must lie in one of these half-planes.■

Note. This is a special case of the *Dirichlet Pigeonhole Principle*: *If we have m objects which lie in n subsets such that $m > n$, then at least one subset must contain at least two of the objects.

2. [*25 points*] (*a*) Let L and M be two lines in the coordinate plane $\mathbb{R}^2$ which meet at a single point. Suppose that a third line N is parallel to L. Show that M and N have a point in common.

(*b*) Suppose we are given $\angle ABC$ in the coordinate plane $\mathbb{R}^2$, and let L be a line in $\mathbb{R}^2$. Prove that L is not contained in the interior of $\angle ABC$. [*Hint:* Try to use part (*a*).]

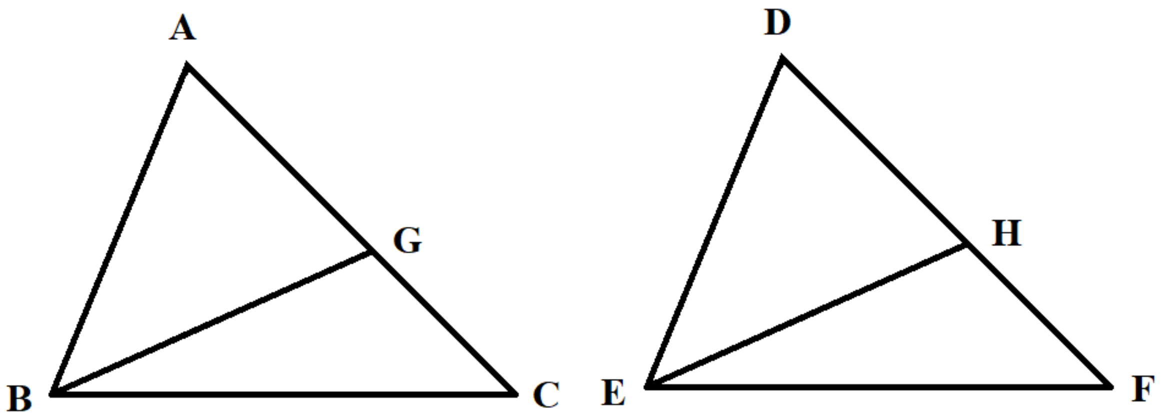
SOLUTION

(*a*) Let X be the point where M and L meet. There is only one line through the common point in $L \cap M$ which is parallel to L; the common point X cannot lie on N because L is assumed to be parallel to N. Since L is parallel to N and M is another line passing through X, the Euclidean Parallel Postulate implies that M is not parallel to N, and hence N has a point in common with M.■

(*b*) If L is contained in the interior then L has no points in common with either AB or BC. Hence L is parallel to both of the lines AB and BC. But both lines pass through the external point B, and the Euclidean Parallel Postulate implies that there is only one line through B which is parallel to L. Therefore L has a point in common with either AB or BC (or both!), which means that L is not contained in the interior of $\angle ABC$.■

3. [*25 points*] Suppose that we are given two triangles $\triangle ABC$ and $\triangle DEF$ in $\mathbb{R}^2$ such that $\triangle ABC \cong \triangle DEF$. Let $G \in (AC)$ and $H \in (DF)$ such that **either** $|\angle ABG| = |\angle DEH|$ **or** $|AG| = |DH|$. Prove that $\triangle GBC \cong \triangle HEF$. [*Hint:* Draw a picture.]

SOLUTION



We have to handle the two hypotheses separately. However, there is one step that both cases have in common: The betweenness conditions $G \in (AC)$ and $H \in (DF)$ and the congruence assumption imply $|AG| + |GC| = |AC| = |DF| = |DH| + |HF|$. Therefore if $|AG| = |DH|$ then we also have $|GC| = |HF|$.

First Case. Assume first that $|\angle ABG| = |\angle DEH|$. Since the original congruence implies $|AB| = |DE|$ and $|\angle ABC = \angle ABG| = |\angle DEF = \angle DEH|$ it follows by ASA that $|\Delta ABG| \cong \Delta DEH$. This implies that $|AG| = |DH|$. As in the preceding discussion, it also follows that $|GC| = |HF|$. Finally, the original congruence implies $|\angle ACB = \angle GCB| = |\angle DFE = \angle HFE|$ and $|BC| = |EF|$, so that $\Delta GBC = \Delta GCB \cong \Delta HFE = \Delta HEF$ by SAS; note that we rearranged the vertices of both triangles compatibly, switching the last two vertices.■

Second Case. Assume now that $|AG| = |DH|$; then the remarks preceding the proof in the first case imply that $|GC| = |HF|$. . Since $G \in (AC)$ and $H \in (DF)$, it follows that G and H are in the interiors of $\angle ABC$ and $\angle DEF$ respectively. and since the original congruence implies $|\angle BAC = \angle BAG| = |\angle EDF = \angle EDH|$ and also $|AB| = |DE|$, it follows that $\Delta BAG \cong \Delta EDH$ by SAS. The latter implies that $|\angle ABG| = |\angle DEH|$, and the latter yields $|BG| = |EH|$. From this we can conclude that $\triangle GBC \cong \triangle HEF$ by SSS.■

4. [*25 points*] The geometric reflection about the line joining $(0,0)$ and $(\cos\theta, \sin\theta)$ is the linear transformation from $\mathbb{R}^2$ to itself has matrix

$$S_\theta \quad = \quad \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$$

and the counterclockwise rotation by an angle of measure α is the linear transformation from $\mathbb{R}^2$ to itself with matrix

$$R_\alpha \quad = \quad \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} .$$

The composite of two reflections $S_\theta \circ S_\varphi$ is equal to a rotation matrix R_α. Express α in terms of θ and φ.

SOLUTION

Use the formulas to compute the matrix product $S_\theta \circ S_\varphi$ explicitly.

$$S_\theta \circ S_\varphi \quad = \quad \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix} \cdot \begin{pmatrix} \cos 2\varphi & \sin 2\varphi \\ \sin 2\varphi & -\cos 2\varphi \end{pmatrix} \quad =$$

$$\begin{pmatrix} \cos 2\theta \cos 2\varphi \ + \ \sin 2\theta \sin 2\varphi & \cos 2\theta \sin 2\varphi \ - \ \sin 2\theta \cos 2\varphi \\ \sin 2\theta \cos 2\varphi \ - \ \cos 2\theta \sin 2\varphi & \sin 2\theta \sin 2\varphi \ + \ \cos 2\theta \cos 2\varphi \end{pmatrix}$$

The trigonometric identities for the sine and cosine of a sum or difference of two angles imply the right hand side is just

$$\begin{pmatrix} \cos 2(\theta-\varphi) & -\sin 2(\theta-\varphi) \\ \sin 2(\theta-\varphi) & \cos 2(\theta-\varphi) \end{pmatrix}$$

and therefore we have $\alpha = 2(\theta - \varphi)$; more precisely, α can be equal to the right hand side plus an arbitrary multiple integral of 2π, but one value is enough for this problem.■