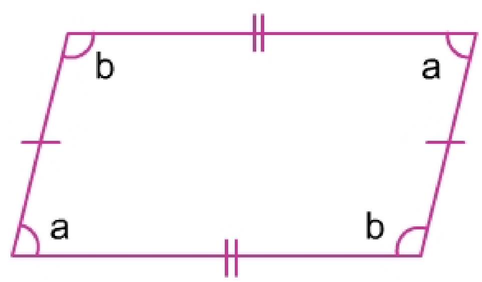
Solutions for `quiz3f20.pdf`

1. The key to proving this is recalling the interrelationships among the various edges and angles in a parallelogram. In addition to the equalities displayed in the drawing below, we also know that $a + b = 180°$.



Suppose now that we are given parallelograms $ABCD$ and $A'B'C'D'$ such that $|AB| = |A'B'|$, $|\angle ABC| = |\angle A'B'C'|$ and $|BC| = |B'C'|$. We need to prove that the remaining 5 corresponding parts of the parallelograms have equal measures. The equalities for the two remaining pairs of sides are implied by the following chains of equations:

$$|CD| \;=\; |AB| \;=\; |A'B'| \;=\; |C'C'| \qquad |AD| \;=\; |BC| \;=\; |B'C'| \;=\; |A'D'|$$

Another chain of equations yields a similar result for one corresponding pair of angles:

$$|\angle CDA| \;=\; |\angle ABC| \;=\; |\angle A'B'C'| \;=\; |\angle C'D'A'|$$

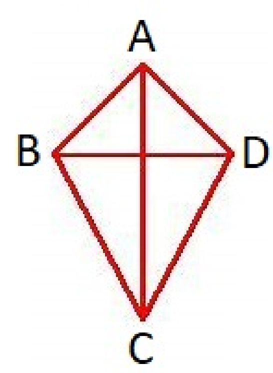
Finally, the remaining corresponding pairs of angles have equal measures by the following chains of equations:

$$|\angle BCD| \;=\; 180° - |\angle CDA| \;=\; 180° - |\angle C'D'A'| \;=\; |\angle B'C'D'|$$

$$|\angle BAD| \;=\; 180° - |\angle ABC| \;=\; 180° - |\angle A'B'C'| \;=\; |\angle B'A'D'|$$

This completes the verification of all eight conditions in the definition of classical congruence.■

2. The following drawing will probably be helpful for understanding the arguments below.



Before considering the two cases, we observe the following: *If the convex quadrilateral $ABCD$ is kite-shaped with respect to AC, then $|\angle ABC| = |\angle ADC|$. Furthermore, $[AC$ bisects $\angle BAD$ and $[CA$ bisects $\angle BCD$.*

The first statement is true because $\triangle ABC \cong \triangle ADC$ by SSS, and it also follows that $|\angle BAC| = |\angle DAC|$ and $|\angle BCA| = |\angle DCA|$. Since a kite is a convex quadrilateral, it follows that C lies in the interior of $\angle BAD$ and A lies in the interior of $\angle BCD$. Therefore we have

$$\begin{aligned} |\angle BAD| &= |\angle BAC| + |\angle DAC| = 2\,|\angle BAC| \\ |\angle BCD| &= |\angle BCA| + |\angle DCA| = 2\,|\angle BCA| \end{aligned}$$

and this proves the assertion about angle bisectors.■

(a) The two kites are congruent in this case. We are given that the four corresponding edges have equal length, and we are also given that $|\angle ABC| = |\angle A'B'C'|$. By the initial observations we immediately have $|\angle ADC| = |\angle A'D'C'|$ as well, for we have the following exactly as in the first exercise:

$$|\angle CDA| = |\angle ABC| = |\angle A'B'C'| = |\angle C'D'A'|$$

To complete the proof we need to compare the measures of the angles whose vertices are (i) A and A', (ii) C and C'.

Just before the beginning of the discussion regarding (a) we noted that $\triangle ABC \cong \triangle ADC$, and similarly we have $\triangle A'B'C' \cong \triangle A'D'C'$. Furthermore, by SAS we have $\triangle ABC \cong \triangle A'B'C'$, and if we combine the observations of this and the preceding sentence we also obtain $\triangle ADC \cong \triangle A'D'C'$. These two congruences yield the following equations involving angle measurements:

$$|\angle BAC| = |\angle B'A'C'|\,,\ |\angle DAC| = |\angle D'A'C'|\,,\ |\angle BCA| = |\angle B'C'A'|\,,\ |\angle DCA| = |\angle D'C'A'|$$

Since we are dealing with convex quadrilaterals, we know that C and C' lie in the interiors of $\angle BDA$ and $\angle B'D'A'$ respectively, and similarly we know that D and D' lie in the interiors of $\angle BCA$ and $\angle B'C'A'$ respectively. If we combine this with the additivity of angle measurements and the previous observations of this paragraph, we obtain the following equations:

$$\begin{aligned} |\angle BAD| &= |\angle BAC| + |\angle DAC| = |\angle B'A'C'| + |\angle D'A'C'| = |\angle B'A'D'| \\ |\angle BCD| &= |\angle BCA| + |\angle DCA| = |\angle D'C'A'| + |\angle B'C'A'| = |\angle B'C'D'| \end{aligned}$$

This completes the verification of all eight conditions in the definition of classical congruence.■

(b) The two kites are also congruent in this case, but the argument is different. We are now given that the four corresponding edges have equal length, and we are also given that $|\angle BAC| = |\angle B'A'C'|$. Since $[AC$ and $[A'C'$ bisect $\angle BAD$ and $\angle B'A'D'$ respectively, it follows that $|\angle BAD| = |\angle B'A'D'|$. Furthermore, since $|AD| = |AB| = |A'B'| = |A'D'|$ we have isosceles triangles $\triangle BAD$ and $\triangle B'A'D'$, and $\triangle BAD \cong \triangle B'A'D'$ by SAS. Therefore $|BD| = |B'D'|$ and

$$|\angle ABD| = |\angle A'B'D'| = |\angle A'D'B'| = |\angle ADB|$$

where the middle equality holds because the triangles are isosceles.

If we now combine $|BD| = |B'D'|$ with the defining conditions for the kites, we also obtain $\triangle DBC \cong \triangle D'B'C'$ by SSS. The two triangles in the preceding expression are isosceles, so we also obtain the following:

$$|\angle DBC| = |\angle D'B'C'| = |\angle B'D'C'| = |\angle BDC|\,, \qquad |\angle BCD| = |\angle B'C'D'|$$

Finally, since kites are convex quadrilaterals we know that B and B' lie in the interiors of $\angle ADC$ and $\angle A'D'C'$ respectively, and similarly we know that D and D' lie in the interiors of $\angle ABC$ and $\angle A'B'C'$ respectively. If we combine the obsevations thus far with the additivity property of angle measurement, we see that

$$|\angle ABC| = |\angle ABD| + |\angle DBC| = |\angle A'B'D'| + |\angle D'B'C'| = |\angle A'B'C'|$$

which means the hypothesis in (a) is true in this case, and by that case we know that the two kites are also congruent under the hypotheses in (b).■