Appendix to Lecture 01

References for background material

The properties of the four basic arithmetic operations in the rational numbers, real numbers and complex numbers can be abstracted into the notion of an **ordered field**. A full list of axioms for an ordered field is given in **Axioms for an Ordered Field (colorado.edu)**. These imply many other identities, some of which can be found in **2. Fields (reed.edu)**. However, the additional key properties of the real number system are less transparent. We shall characterize the real numbers informally; one can prove that our characterization is equivalent to the usual ones which appear in upper level undergraduate mathematics texts.

FACT. The real numbers are an ordered field containing the rational numbers with the following additional properties:

1. If real numbers x and y satisfy $x < y$, then there is a rational number r such that $x < r < y$.

2. The real numbers are a maximal ordered field satisfying the preceding conditions.

This is enough to show that every real number between 0 and 1 has a decimal expansion of the form

$$a_1/10 + a_2/10^2 + + a_3/10^3 + \ldots \quad \text{(finite or infinite sum)}$$

where each a_k is an integer such that $0 \leq a_k \leq 9$ and the only time such a expansions are not unique is when one expansion has the form

$$a_1/10 + a_2/10^2 + + a_3/10^3 + \ldots + a_m/10^m \text{ (finite sum, } a_m > 0)$$

and the other has the form

$$a_1/10 + a_2/10^2 + + a_3/10^3 + \ldots + (a_m\text{-}1)/10^m + 9/10^{m+1} + 9/10^{m+2} + \ldots$$

(all nines from **m + 1** onwards). Centuries of experience show that all of this functions very well computationally, but working with it conceptually, and checking that we get an equivalent system if we replace 10 with another computational base, turns out to be very challenging in some respects. Effective ways of avoiding these difficulties were not discovered until the 19th century.

Finally, the complex numbers are obtained from the real numbers by adjoining a number **i** whose square is equal to – 1.

Set theoretic background. More than enough background in set theory can be found in **1. Notation, Undefined Concepts and Examples (reed.edu)**.