Undefined concepts

Euclid tried to define everything — not formally possible

2D geometry — Assume given a set P, call its members points. (Nonempty!)

Also assume we are given a family of subsets we shall call lines. $\mathcal{L}$ = proper subsets

Assume the following axioms (Incidence)

(I1) Given two distinct points, there is a unique line containing them.

(I2) Every line contains ≥ 2 points.

Models for axioms Ideally, ordinary geometry

Also, S = set with ≥ 2 elements, $\mathcal{L}$ = all subsets with exactly two elements.

... and many more...

NOTATION xy = line cont. x & y

Only one noteworthy theorem.

If $L \neq M$ are lines in P, then $L \cap M$ has at most one point.

Proof Say $x \neq y$, both in $L \cap M$.

By I3 there is only one line N so $x, y \in N$. But L & M have this property. So must have $L = M$.

COLLINEAR SET A, $A \subseteq L$, some line.

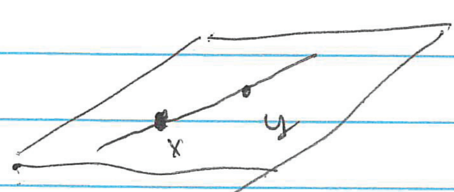
3D geometry S nonempty set
$\mathcal{L}$ = proper subset family
$\mathcal{P}$ = ~~a disjoint~~ another one, disjoint from $\mathcal{L}$

First two axioms plus

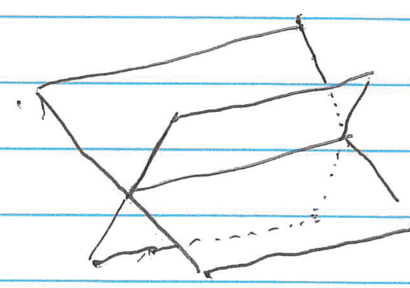
(I3) Given a noncollinear subset $\{x, y, z\} = B$, there is a unique plane P containing B.

(I4) If x, y distinct in P, then $xy \subseteq P$.

(I5) If P and Q are planes such that $P \neq Q$ then $P \cap Q$ is a line.



I4

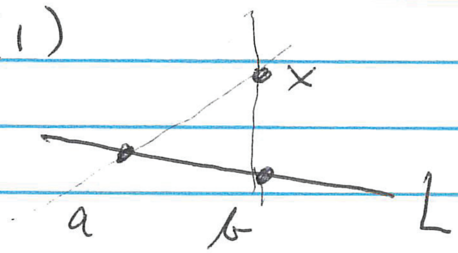

I5

(I6) Every plane has ≥ 3 points.

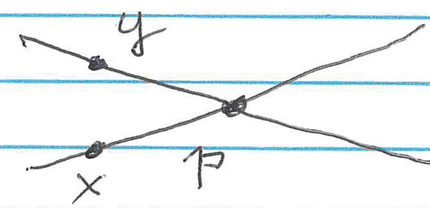
<u>Two consequences</u> (1) Given line L and $x \notin L$, there is a unique plane P so $L \subseteq P$ and $x \in P$.

(2) If two distinct lines meet at a single pt., then they determine a plane.

(1)



(2)



Pictures make logic transparent, but are not proofs themselves!

Now we need to add more data

<u>Distance Postulates</u>
<u>Ruler Postulate</u>

<u>Distance</u> $d : S \times S \longrightarrow [0, \infty)$ given

$d(x,y) \geq 0$, equality $\Longleftrightarrow$ $x = y$
$d(x,y) = d(y,x)$. [No triangle inequality yet!]

<u>Ruler</u>. There are 1-1 correspondences

$f : L$ (each line) $\longleftrightarrow \mathbb{R}$ so that
$$d(x,y) = |f(x) - f(y)|.$$

(Placement)

Strong Ruler Property

Given $x \neq y$ in L, can find f so that $f(x)=0$, $f(y)>0$.

How to derive this conclusion. Take any f_0.
Let $\varepsilon_0 = 1$ if $f_0(x) < f_0(y)$, $\varepsilon_1 = -1$ otherwise.
Check that $g(t) = \varepsilon_0 [f_0(t) - f_0(x)]$ is 1-1 and $d(t_1, t_2) = |g(t_1) - g(t_2)|$.

1-1: $g(t_1) = g(t_2) \Rightarrow \varepsilon_0 [f_0(t_1) - f_0(x)] = \varepsilon_0 [f_0(t_2) - f_0(x)]$
$\Rightarrow f_0(t_1) - f_0(x) = f_0(t_2) - f_0(x) \Rightarrow f_0(t_1) = f_0(t_2) \Rightarrow$
$t_1 = t_2$ since f_0 is 1-1.

onto: Let $a \in \mathbb{R}$. Want $a = [f_0(t) - f_0(x)]\varepsilon_0$ some t.
Well, $\varepsilon_0 a = f_0(t) - f_0(x)$, $\varepsilon_0 a + f_0(x) = f(t)$
suggests the right choice of t. Substitute (work backwards) to verify that $a = g(t)$.

distance preserving: $d(t_1, t_2) = |f_0(t_1) - f_0(t_2)|$ is given.

R.H.S. $= |[f_0(t_1) - f_0(x)] - [f_0(t_2) - f_0(x)]|$

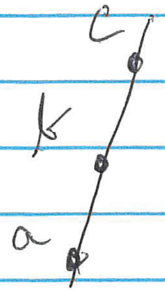
and $|\varepsilon_0| = 1$ means this is

$|\varepsilon_0 ([f_0(t_1) - f_0(x)] - [f_0(t_2) - f_0(x)])| =$

$|\varepsilon_0 [f_0(t_1) - f_0(x)] - \varepsilon_0 [f_0(t_2) - f_0(x)]| =$

$|g(t_1) - g(t_2)|$.

Why do we need this?



Given three points on a line, we "see" that one is between the other two. This was only discussed casually in Euclid, but it is absolutely essential for a logically sound treatment of classical geometry.

RECALL When is $|a+b| = |a| + |b|$ in $\mathbb{R}$?

Either $a, b \geq 0$ or $a, b \leq 0$.

Define $a * b * c$ (b is between a & c) $\iff$

$d(a,c) = d(a,b) + d(b,c)$.

This gives us all we need. The following properties are "obvious" but must be verified.

Betweenness and ruler functions

Theorem On line L with ruler function f,

$$a * b * c \iff f(a) < f(b) < f(c) \text{ or }$$

$f(a) > f(b) > f(c)$.

Derivation $|p+q| = |p| + |q| \iff$

$p, q \geq 0$ or $p, q \leq 0$. Now let $p = f(a) - f(b)$ & $q = f(b) - f(c)$, so $p+q = f(a) - f(c)$ and $|p+q| = |p| + |q| \iff d(a,c) = d(a,b) + d(a,c)$. The case $p, q \geq 0$ corresponds to $f(a) > f(b) > f(c)$, and $p, q \leq 0$ corresponds to $f(a) < f(b) < f(c)$. ∎

Theorem Given p, q & r distinct on L, one & only one is between the other two.

Proof Six cases

$f(p) < f(q) < f(r)$	$f(q) < f(r) < f(p)$	$f(q) < f(p) < f(r)$
$f(p) > f(q) > f(r)$	$f(q) > f(r) > f(p)$	$f(q) > f(p) > f(r)$
q between	r between	p between

See lecture02a.pdf for a cleaned up version of this material.

The following are typical applications which are needed for more substantial results.

(1) $a*b*c$ & $b*x*c \Rightarrow a*x*c$

First Find ruler fcn so $f(a) < f(b)$.

By previous, have $f(a) < f(b) < f(c)$.

Now $b*x*c \Rightarrow$ either $f(b) < f(x) < f(c)$ or $f(b) > f(x) > f(c)$. Second violates previous conclusion, so $f(a) < f(b) < f(x) < f(c)$.

(2) $a*x*c$ & $a*y*c \Rightarrow a*x*y$ or $a*y*x$.

Choose ruler $f(a) = 0$, $f(c) > 0$.

$0 < f(x) < f(c)$
$0 < f(y) < f(c)$ Why?

So either $0 < f(x) < f(y)$ or $0 < f(y) < f(x)$.

Dictionary

$A \neq B$, f ruler with ~~$f(a) = 0$~~, $f(b) > f(a)$

closed seg't	$[AB]$	$x = a, b$ or $a*x*b$	$f(x) \in [f(a), f(b)]$
open seg't	(AB)	$a*x*b$	$f(x) \in (f(a), f(b))$
closed ray	$[AB$	$x = a$, $a*x*b$, $x = b$, $a*b*x$	$f(x) \in [f(a), \infty)$
open ray	$(AB$	$x \in [AB - \{A\}$	$f(x) \in [-\infty, f(a)]$
opp closed ray	$[AB^{op}$	$x = A$ or $x \in L - (AB$	$f(x) \in (-\infty, f(a)]$
opp ~~closed~~ open ray	$(AB^{op}$	~~$x \in [AB^{op} - \{A\}$~~ $x \in L - [AB$	$f(x) \in (-\infty, f(a))$.