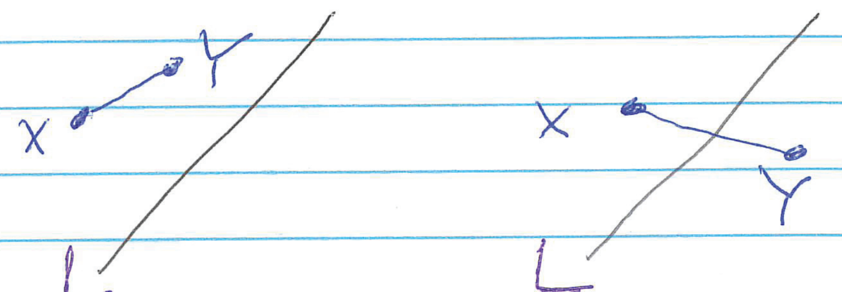
# Plane separation

A concept related to betweenness, also discussed casually in Euclid but requires a logically sound treatment.

Plane $P$, line $L \subseteq P$, $X$ & $Y$ in $P$ but not $L$




same side of L     opposite sides of L

Notice the second holds $\Leftrightarrow$ the lines $L$ and $XY$ meet at a point between $X$ & $Y$.

Plane Separation Postulate   $L$ & $P$ as above. Then $P - L$ is a union of two nonempty disjoint sets $A$ & $B$ such that

(1) If $X$ & $Y$ in either $A$ or $B$, then $(XY) \subseteq$ the appropriate set ($A$ & $B$ are convex)

No dents or holes in a convex set

convex


neither convex

(2) $X \in A$ & $Y \in B \Rightarrow L \cap (XY) \neq \phi$.

<u>Space separation postulate</u> Similar, but replace $\left\{\begin{matrix}\text{line}\\ \text{plane}\end{matrix}\right\}$ with $\left\{\begin{matrix}\text{plane}\\ \text{space}\end{matrix}\right\}$.

The second implies the first for all planes in the 3-space (not to be used very much).

Algebra yields line separation: $L$ line, $X \in L$. Then $L - \{X\} =$ two disjt open rays $(AB$ & $(AC$ s.t. (1) $X, Y$ on one open ray $\Rightarrow (XY) \subseteq$ open ray
(2) $X$ in one, $Y$ in other $\Rightarrow X * A * Y$.
(Use ruler function!)

<u>Theorem</u> $L$ & $M$ lines meeting at $X$, $Y$ & $Z$ in $M - \{X\}$.
(1) $Y, Z$ on same side of $L \Leftrightarrow$ either $X * Y * Z$ or $X * Z * Y$.
(2) $Y, Z$ on opposite sides $\Leftrightarrow Y * Z * X$.

<u>Verification</u> One of $X, Y, Z$ is between the other two (and only one). Enough to check (2) is true.

($\Longrightarrow$) True by Plane Separation Postulate.

($\Longleftarrow$) Suppose on same side. Which of

$X*Z*Y$, $X*Y*Z$, $Z*X*Y$ is true?

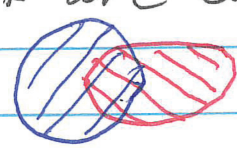
Why not the last one? $Z, Y$ on same side

$\Rightarrow (ZY) \subseteq$ that side $\Rightarrow X$ is too. But

the side is contained in $L - \{X\}$, contradiction.

<u>Theorem.</u> If $A$ & $B$ are convex, so

is $A \cap B$.



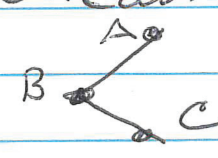
<u>Proof</u> Let $X, Y \in A \cap B$.

$$\left.\begin{array}{l} \text{Then } X, Y \in A \Longrightarrow (XY) \subseteq A \\ \text{—''—}\ B \Longrightarrow (XY) \subseteq B \end{array}\right\} \text{Hence } (XY) \subseteq A \cap B.$$

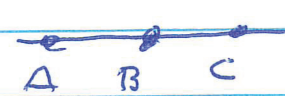
Suffices to define other regions (convex).

<u>ANGLES.</u> $A, B, C$ noncollinear.

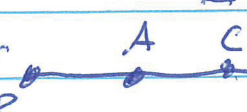
$\measuredangle ABC = \overline{[BA} \cup \overline{[BC}$




For us, straight angles



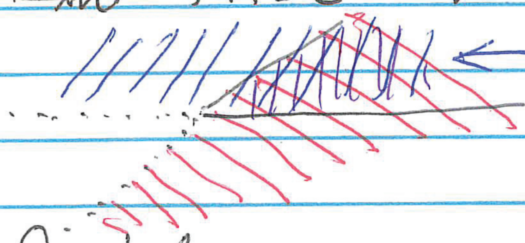

and degenerate angles




are <u>NOT</u> angles!!

## Interior of an angle

$\text{Int}\,\angle ABC = A\text{-side } BC \cap C\text{-side } AB$



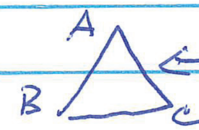
← interior (edges not included!)

Simple consequence. If $X \in AC$, then $X \in \text{Int}\,\angle BAC$.

Proof. $A * X * C \Rightarrow X$ & $A$ on same side $BC$
$X$ & $C$ on same side $AB$.

Exercise Same conclusion if $Y \in (BX$.
($Y$ & $X$ same side of $BC$ and $AB$)

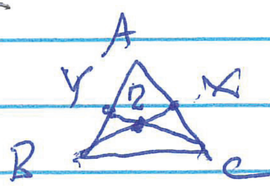
Interior of triangle $\triangle ABC = [AB] \cup [BC] \cup [AC]$




$\text{Int}\,\triangle ABC =$

$\text{Int}\,\angle ABC \cap \text{Int}\,\angle BCA \cap \text{Int}\,\angle CAB$

Exercise It's the intersection of any two.

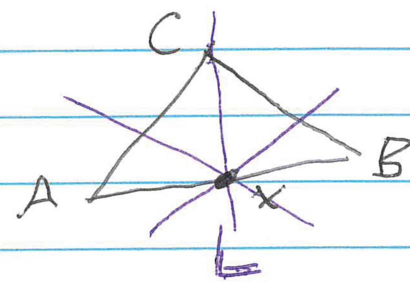



$A * X * C$
$A * Y * B$
$Z \in (CX) \cap (BY)$

To prove $Z \in \text{Int}\,\triangle ABC$.

## Two more substantial results IN THE PLANE

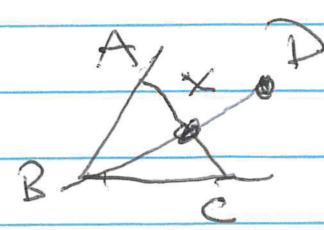
**Pasch's Theorem** Given $\triangle ABC$ and $L$ such that $L \cap (AB) = \{X\}$. Then one of the following is true: (1) $L \cap (BC) \neq \phi$, (2) $C \in L$ (3) $L \cap (AC) \neq \phi$




Used in Euclid w/o assuming it.

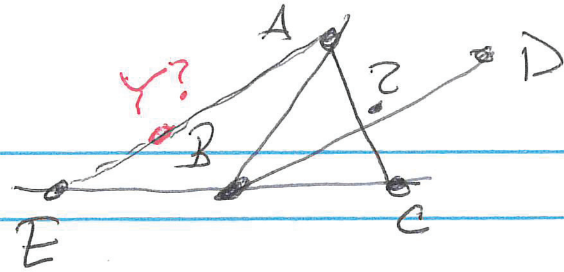
Proof. Three options for C: (on opp side B at L) (same side of L as B) $C \in L$. Analyze first two cases. Opp sides: $(BC) \cap L \neq \phi$ by Plane Separation. Same side. B & A on opp sides since $B * X * C$. Hence C & A on opp sides also. But then $L \cap (AC) = \phi$.

**Cross bow Theorem.**




Given $D \in \text{Int} \measuredangle ABC$. Then there is a point $X \in (BD \cap (AC)$.

Proof.




E*B*C

Then BD meets either (AC) or (AE].

Note $A \notin BD$ since A, B, D not collinear by hyp.

Need to show $BD \cap (AE) = \emptyset$. Say $Y \in BD \cap (AE)$. Then E, Y on same side of AB.

So Y on opp side AB as D + C ($D \in$ Interior!)

Hence (YD) meets AB, **INSERT: and since Y, B, D are collinear, the common point must be B** so D & Y on opp sides of BC. However $A*Y*E \Rightarrow$ Y, A, D all on same side of BC! Contradiction. Source? Assuming $BD \cap AE \neq \emptyset$. Hence $BD \cap (AC) \neq \emptyset$. Let X be this pt.

Finally, must show $X \in (BD$.

$A*X*C \Rightarrow$ A & X on same side BC

C & X on same side AB

But also A+D on same side BC, so likewise for D and X. By previous warmups, $X \in (BD$.

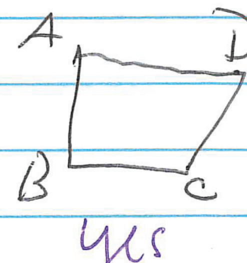
First, try to understand passively, then actively as much as possible!

## Diagonal Theorem.

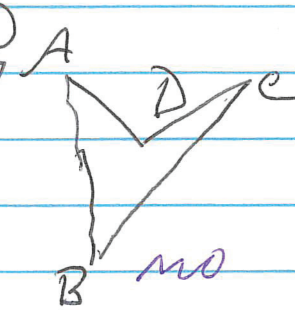
A, B, C, D no 3 collinear

Say A, B, C, D form the vertices of a convex quadrilateral if:

A, B on same side CD
B, C ——"—— AD
C, D ——"—— AB
D, A ——"—— BC

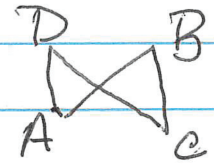



yes




no

Two edges disjt. or have ONE common endpt.

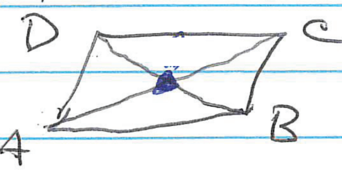



also no.

$\square ABCD = [AB] \cup [BC] \cup [CD] \cup [AD]$

Note $\square ABCD$ is not a convex set!

Thm. Given $\square ABCD$, there is a point in $(AC) \cap (BD)$.




Proof. Use Crossbar Thm. twice.

Hypotheses $\Longrightarrow C \in \text{Int} \measuredangle DAB = $ D-side AB $\cap$ B-side AD

So $(AC$ meets $(BD)$ in some point X.

Also $(BD$ meets $(AC)$ in some point Y.

Now $AC \neq BD$ by def of convex quad.

Hence $AC \cap BD$ has at most one point.

Since $X, Y \in AC \cap BD$, must have $X = Y$ and $X \in (AC)$.