Math 133
Fall 2021

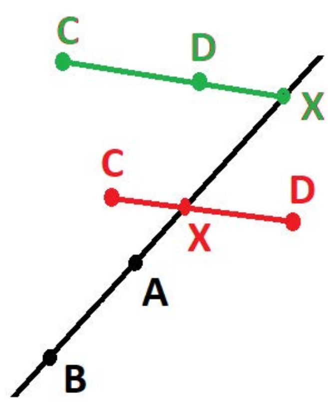
SOLUTIONS FOR WEEK 02

Assume that $(\mathbf{S}; \mathcal{P}; \mathcal{L}; d; \alpha)$ or $(\mathbf{P}; \mathcal{L}; d; \alpha)$ is a system which satisfies the Incidence, Ruler, Plane or Space Separation (as appropriate), Angle Measurement and Triangle Congruence Axioms. Here α denotes the angular measurement function $\alpha(\angle ABC)$ which we generally denote by $|\angle ABC|$, and recall that the distance $d(A, B)$ is frequently denoted by $|AB|$.

1. Let $\mathbf{H}$ and $\mathbf{K}$ denote the two sides of L in P. Then the hypotheses imply that $A \in \mathbf{H}$ and $B \in \mathbf{K}$ or vice versa. The proof in the second case follows from the argument in the first case if we switch the roles of $\mathbf{H}$ and $\mathbf{K}$, so it suffice to prove the result when the first alternative holds. Since the hypotheses now imply that $C \in \mathbf{H}$, it follows that A and C lie on the same side of L.■

2. As in the previous exercise, let $\mathbf{H}$ and $\mathbf{K}$ denote the two sides of L in P. Then either $A \in \mathbf{H}$ or $S \in \mathbf{K}$, and much as before we might as well assume the first alternative holds. Then the hypotheses first imply that $B \in \mathbf{H}$, and from this we may also conclude that $C \in \mathbf{H}$. Therefore all three points lie on the same side of L.

3. The third option holds, and the following examples show this. If we take C, D so that $X * C * D$ for some $X \in AB$, then C and D lie on the same side of AB, but if take C, D so that $C * X * D$ for some $X \in AB$, then C and D lie on opposite sides of AB.■



4. By definition, the interior of the triangle is

$$\text{Int}\,\angle BAC \;\cap\; \text{Int}\,\angle ABC \;\cap\; \text{Int}\,\angle ACB$$

and by the definition of angle interior we may write this as an intersection of the following half-planes:

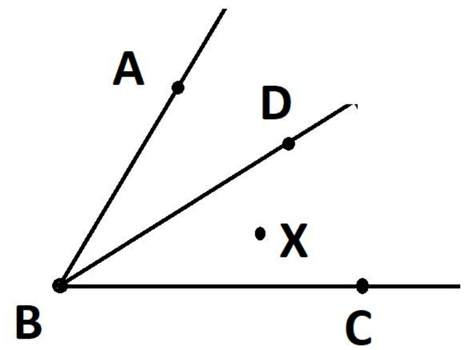
$$(A-\text{side } BC) \cap (C-\text{side } AB) \cap (A-\text{side } BC) \cap (B-\text{side } AC) \cap (A-\text{side } CB) \cap (B-\text{side } CA)$$

This simplifies to $(A - \text{side } BC) \;\cap\; (C - \text{side } AB) \;\cap\; (B - \text{side } AC)$ after removing redundancies. However, we also obtain the same intersection if we simplify Int $\angle BAC \cap$ Int $\angle ACB$, so the interior of the triangle equals the intersection of the interiors of the two specified angles.■

5. Suppose that $\triangle ABC$ is equilateral, so that $|AB| = |BC| = |AC|$. Two applications of the Isosceles Triangle Theorem imply that $|\angle ABC| = |\angle ACB|$ and $|\angle CBA| = |\angle CAB|$. Since $\angle CBA = \angle ABC$ it follows that the triangle is also equiangular.

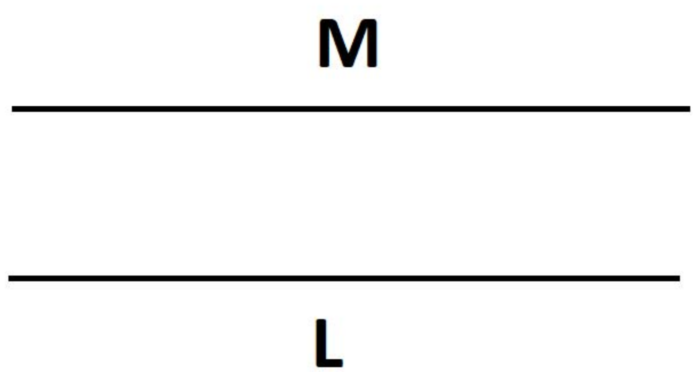
Conversely, if $\triangle ABC$ is equiangular, so that $|\angle BAC| = |\angle ACB| = |\angle ABC|$, then two applications of the Isosceles Triangle Theorem imply that $|BC| = |BA|$ and $|AC| = |BC|$, so that the triangle is equilateral.■

6. Suppose that $X \in \text{Int}\,\angle DBC = (D - \text{side } BC) \cap (C - \text{side } BD)$. By the definition of angle interiors, it suffices to prove that $X \in A - \text{side } BC$.



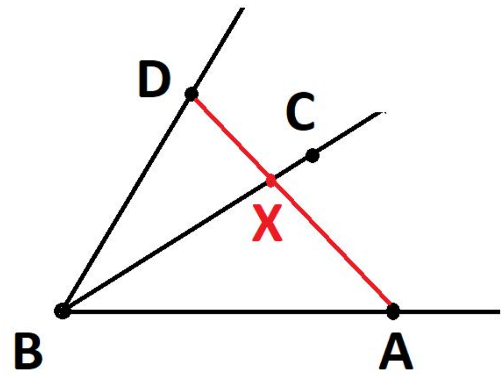
By the Crossbar Theorem there is a point $Y \in (DC) \cap (BX$. Since $Y \in (BX$ we know that Y and C lie on the same side of BC, and since $Y \in (DC)$ we know that Y and D also lies on the same side of BC. Now the hypotheses imply that A and D lie on the same side of BC, and by a previous exercise it follows that A, D, X, Y all lie on the same side of BC.■

7. Suppose that $X, Y \in M$ lie on opposite sides of L. Then by Plane Separation there is a point $Z \in XY \cap L = M \cap L$. This proves the contrapositive to the exercise (and hence yields a proof of the exercise itself).■



8. Repeated applications of the preceding exercise show that A and B are on the same side of CD, B and C are on the same side of AD, C and D are on the same side of AB, and A and D are on the same side of BC. Therefore A, B, C, D (in that order!) form the vertices of a convex quadrilateral.■

9. We are given that C and D lie on the same side of AB but A and D lie on opposite sides of BC. It will suffice to prove that C and A lie on the same side of BD.



Since A and D lie on opposite sides of BC, we know that there is a point $X \in (AD) \cap BC$. It follows that A and X lie on the same side of BD. By the hypotheses we also know that C and D lie on the same side of AB, and therefore C and X lie on the same side of AB. By the results on plane separation, this means that $X \in (BC = BC \cap (C - \text{side } AB)$. Furthermore, we also have $(BC = (BX$, so that C and X lie on the same side of BD. Combining this with the conclusion of the first sentence in the paragraph, we see that $C \in (A - \text{side } BD)$, which is what we needed to prove.∎

10. If CA meets BD at a point X, then both lines are perpendiculars to AB which pass through X. Since there is only one such perpendicular, we have $CA = BD$, so that A, B, C, D are collinear. This contradicts the hypothesis, and the only remaining possibility is that $CA \cap BD = \emptyset$.∎