

Inequalities for triangles

"Locus Theorem" $A \neq B$ in plane P . Then the set of all X in P so $AX = BX$ is the line $L \perp AB$ so that $\text{midpt}(A, B) \in L$.

classical
"locus"

$A \neq X \neq B$ and $|AX| = |XB|$.

Exists by Ruler Postulate

= modern

"set"

Proof ^{WANT} $|AX| = |BX| \Leftrightarrow X \in L$.

Case 1 $X \in AB$. Take ruler f so $f(B) > f(A)$.

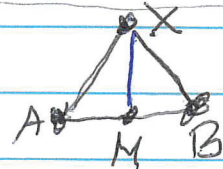
Then $|AX| = |BX| \Leftrightarrow |f(B) - f(X)| = |f(A) - f(X)| \Leftrightarrow$
Squares are equal (since quantities ≥ 0). \Leftrightarrow

$$f(X)^2 - 2f(B)f(X) + f(B)^2 =$$

$$f(X)^2 - 2f(A)f(X) + f(A)^2 \quad \Leftrightarrow$$

$$f(B)^2 - f(A)^2 = 2f(X)[f(B) - f(A)] \quad \Leftrightarrow \text{algebra (why?)}$$

$$\frac{1}{2}(f(B) + f(A)) = f(X).$$



M midpt.

Case 2 $X \notin AB$.

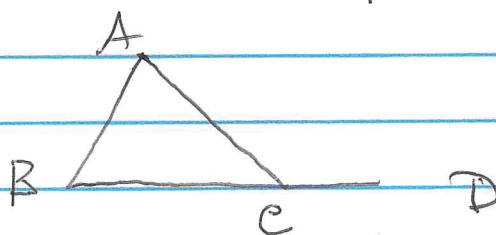
If $|XA| = |XB|$, then $\triangle XMA \cong \triangle XMB$
by SSS $\Rightarrow |XMA| = |XMB|$.

But $A * M * B \Rightarrow 180 = |\angle XMA| + |\angle XMB| = 2|\angle XMA|$
 $\Rightarrow |\angle XMA| = 90$

Conversely, if $|\angle XMA| = 90$ then $\triangle XMA \cong \triangle XMB$ by SAS and hence $|XA| = |XB|$.

Exterior Angle Theorem (importance often not stressed).

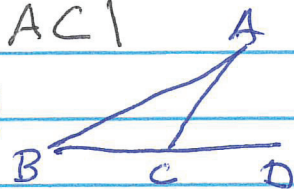
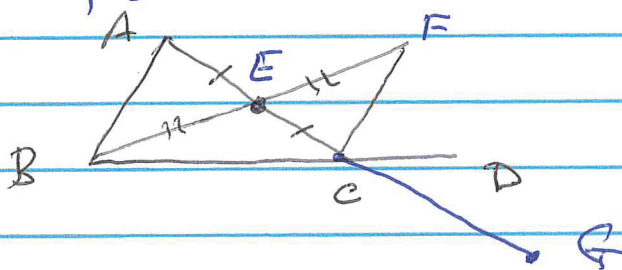
$\triangle ABC$ +
 $B * C * D$.



Then $|\angle ACD| > |\angle ABC|, |\angle BAC|$



Can't expect $|\angle ACD| > |\angle ACB|$



$A * C * G$

Proof Construct E, F, G as illustrated:

$E = \text{midpt. } [AC], B * E * F \text{ and } |BF| = 2|BE|,$
 so $|BE| = |EF|$.

Then $|\angle FEC| = |\angle BEA| = |\angle CEA|$, so
 $\triangle EFC \cong \triangle EAB$ by SAS. Hence

$|\angle EFG| = |\angle FCE| = |\angle EAB| = \angle BAE$.

Looks like $\angle FCE < \angle DCE$, will follow if $F \in \text{Int } \triangle ECD = \triangle AED$,

$B * E * F + A * E * C \Rightarrow F \in A\text{-side } BC$.

$B * C * D + B * E * F \Rightarrow$ Flip F, D on op B -side AC , so $F \in D\text{-side } AC$

These show $F \in \text{Int } \triangle ACD$ & hence

$$\angle ACD > \angle ACF = \angle ECF = \angle BAC.$$

Must now show $\angle ACF > \angle ABC$.

Same idea, switch roles of A & B , replace

D by G , note $\angle ACD = \angle BCF$ by Vert. Angle Thm.

Cor. $\angle ACB + \angle BAC < 180$.

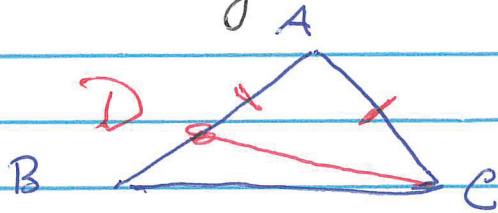
Proof. First show $A \in \text{Int } \triangle BCF$.

$A * E * C + B * E * F \Rightarrow A$ & F on same side BC . Also A & B on same side CF .

Hence $180 \geq \angle BCF = \angle BCA + \angle ACF = \angle BCA + \angle BAC$.

Isosceles Δ Thm. says that in ΔABC ,
 $|AB| \neq |AC| \Leftrightarrow |\angle ABC| \neq |\angle ACB|$. In fact
Larger side is opposite larger angle.

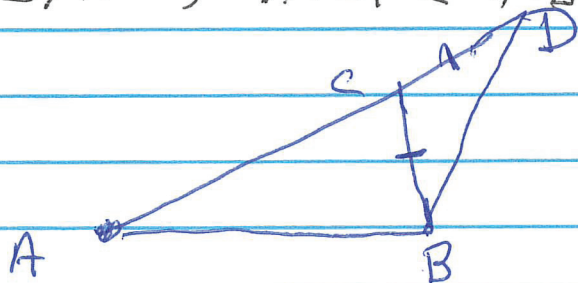
$|AB| > |AC| \Rightarrow |\angle ACB| > |\angle ABC|$
 (and conversely - switch roles of B & C).



Proof. Have $D \in (AB)$ so $|AD| = |AC|$, and
 in fact $A \neq D \neq B$ since $|AD| = |AC| < |AB|$
~~in fact~~ $|\angle ACB| > |\angle ACD|$ since
 $D \in \text{Int } \angle ACB$. $|\angle ACD| = |\angle ADC|$ by
 Isosceles Δ Thm. $|\angle ADC| > |\angle ABC|$ by
 Ext. \angle Thm. Combine these to get $|\angle ACB| > |\angle ABC|$.

THE TRIANGLE INEQUALITY

In ΔABC , $|AC| < |AB| + |BC|$.



Proof. Locate D so $A \neq C \neq D$ and $|CD| = |CB|$.

Then $C \in \text{Int}(\triangle ABD)$ and

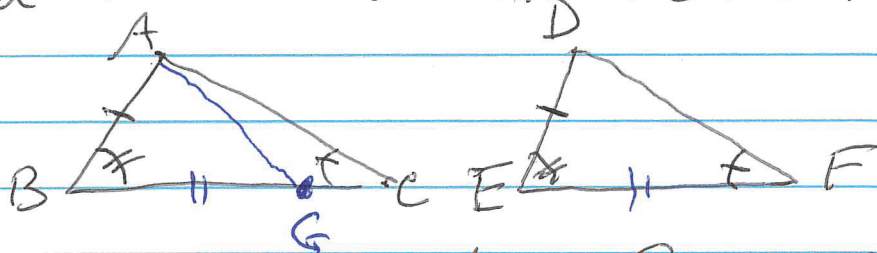
$|\triangle ADB| = |\triangle CDB| < |\triangle ADB|$. Hence

$|AD| > |AB|$. But $|AD| = \cancel{|AC|} |AC| + |CD| =$

$|AB| + |BC|$, so this is greater than $|AC|$.

A.A.S. Congruence In $\triangle ABC$ & $\triangle DEF$

if $|\triangle ACB| = |\triangle DFE|$, $|\triangle ABC| = |\triangle DEF|$
and $|AB| = |DE|$, then $\triangle ABC \cong \triangle DEF$.



Does
not
use
anything
about
angle sum
of \triangle !

Proof Suppose not. Then either

$|BC| < |EF|$ or $|EF| < |BC|$. Switching
roles of \triangle s, may as well assume 2nd one.
otherwise \cong by SAS.

Take $G \in BC$ so $|BG| = |EF|$. Then

$|\triangle ABG| \cong |\triangle DEF|$ by SAS, and hence

$|\triangle AGB| = |\triangle DFE| = |\triangle ACB|$. But Ext \triangle thus

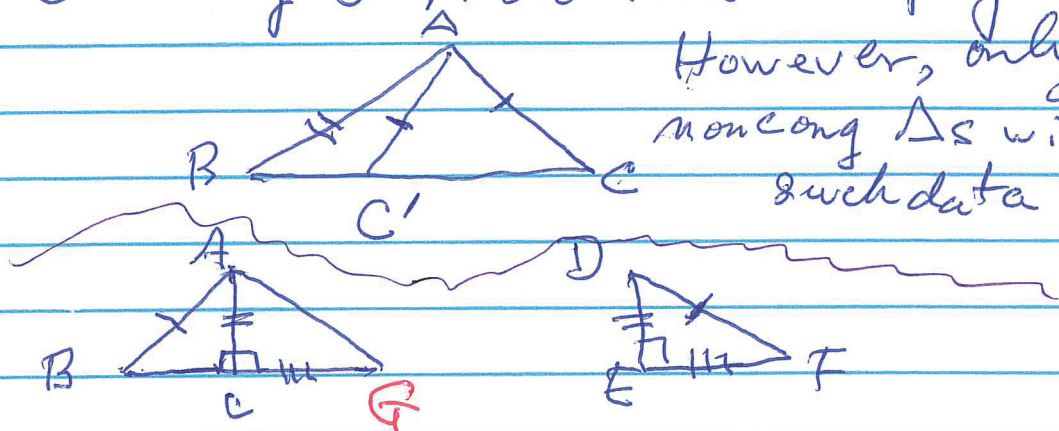
implies $|\triangle AGB| < |\triangle ACB|$. Contradiction. Hence
 $|BC| = |EF|$.

Cor Hypotenuse-angle for rt. Δ s. \cong

Thm. Hypotenuse-side \cong for rt Δ s.

Usually SSA does not imply \cong

However, only two noncong Δ s with such data.



Proof Construct G so $B * C * G$ and $|CG| = |EF|$ and $\Delta ACG \cong \Delta DEF$ by SAS.

Then $|DF| = |AG|$ and $|AB| = |DF| \Rightarrow$

$|AB| = |AG| \Rightarrow$ ~~isosceles~~

$|AB| = |AG| \Rightarrow \Delta ABC = \Delta ACG$ by Isosceles Δ Thm.

But $|ACG| = |DFE|$. Hence $\Delta ABC \cong \Delta DFE$ by A.A.S.

Does not use Pythagorean Theorem.
(as is often the case in H.S. textbooks).

System satisfying
 $(P; L, d, \alpha)$ or
 $(S; P, L, d, \alpha)$ is called
a neutral geometry.