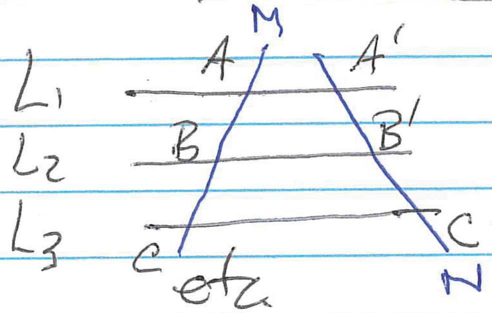
Parallel Projection



L_i mutually $\parallel$

M & N transversals

First observation $A*B*C \Rightarrow A'*B'*C'$.

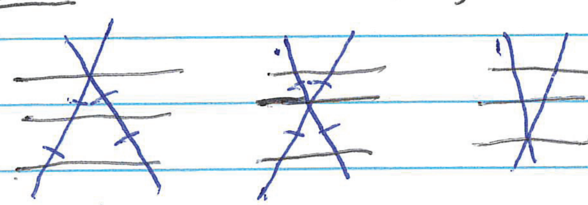
Proof B lies in strip between L_1 & L_3 (see HW).

$B \in L_3$-side $L_1 \cap L_1$-side L_3 (def of strip)

Either $B' = B$ or $BB' = L_2$. In either case, $B' \in$ [previous] $\Rightarrow A'*B'*C'$.

Prop 1 $|AB| = |AC| \Rightarrow |A'B'| = |A'C'|$.

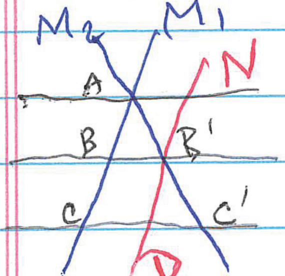
Proof Cases $A = A'$, $B = B'$ or $C = C'$



1st & 3rd same up to switching roles of variables $A \leftrightarrow C$, $M_1 \leftrightarrow M_2$ etc.

Main case $M_1 \nparallel M_2$

Do $M_1 \parallel M_2$ at end.



$A = A'$

Take N through B' $N \parallel M_1$

$|\measuredangle ABB'| = |\measuredangle ACC'|$ (corresp $\measuredangle$s)

$|\measuredangle AB'B| = |\measuredangle AC'C|$

$D \in N \cap L_3$

A & C' on opp sides N

A & C on same side N

$\Rightarrow$ C & C' on opp sides N.

hence $C*D*C'$

Also recall $|\angle AB'B| = |\angle AC'C|$ by remarks on prev. page.

This implies B, B', D, C vertices of ▱, and $\measuredangle ACC' = \measuredangle ACD$ and $\measuredangle B'DC'$ are corresp $\measuredangle$s.

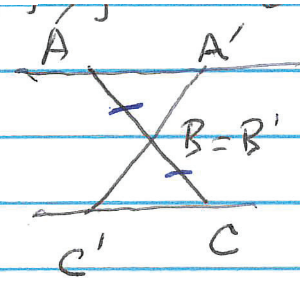
Hence $|AB| = |B'D|$ $(= |BC|)$ and

$|\measuredangle ACC'| \overset{\text{(asa)}}{=} |\measuredangle ABB'| = |\measuredangle C'B'D|$. Hence

$\Delta ABB' \cong \Delta B'DC'$, so that

$|A'B'| = |B'C'|$.

Case $B = B'$



$|AB| = |BC|$ given

$|\measuredangle ABA'| = |\measuredangle CBC'|$ Vertical angles

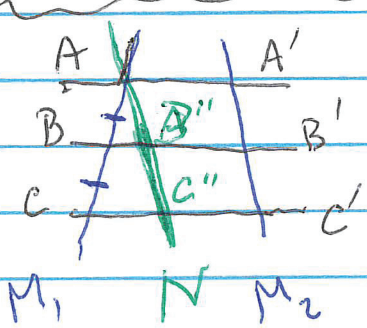
$|\measuredangle AA'C| = |\measuredangle CC'A|$ Alternate Interior Angles.

So $\Delta ABA' \cong \Delta CBC'$ by ASA, and hence

$|A'B = A'B'| = |C'B = C'B'|$.

Case $\{A, B, C\} \cap \{A', B', C'\} = \emptyset$.

$\left\{\begin{array}{l}\text{no condition} \\ \text{whether} \\ M_1 \parallel M_2 \text{ or NO.}\end{array}\right.$



Choose $N \parallel M_2$ so $A \in N$

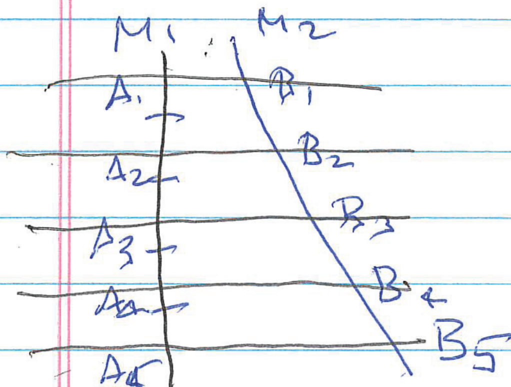
Then ~~By~~ $N \neq M_2$ otherwise $A \in M_2$, which is not the case

Previous reasoning shows $|AB''| = |B''C'|$. But now we have ▱s $AA'B''B$ + $B''B'C''C$, so $|AB''| = |A'B'|$, $|B''C''| = |B'C'|$ and finally $|A'B'| = |AB''| = |B''C''| = |B'C'|$.

mutually ∥

<u>Notebook Paper Theorem</u> L_i, M_j as before with $A_i \in L_i \cap M_1$, $A_1 * A_2 * A_3 \cdots * A_m$.

$B_i \in L_i \cap M_2$. Then $B_1 * \cdots * B_m$. Furthermore, if $|A_{i+1}A_i| = |A_2A_1|$ for all i, then $|B_{i-1}B_i| = |B_2B_1|$ for all i.

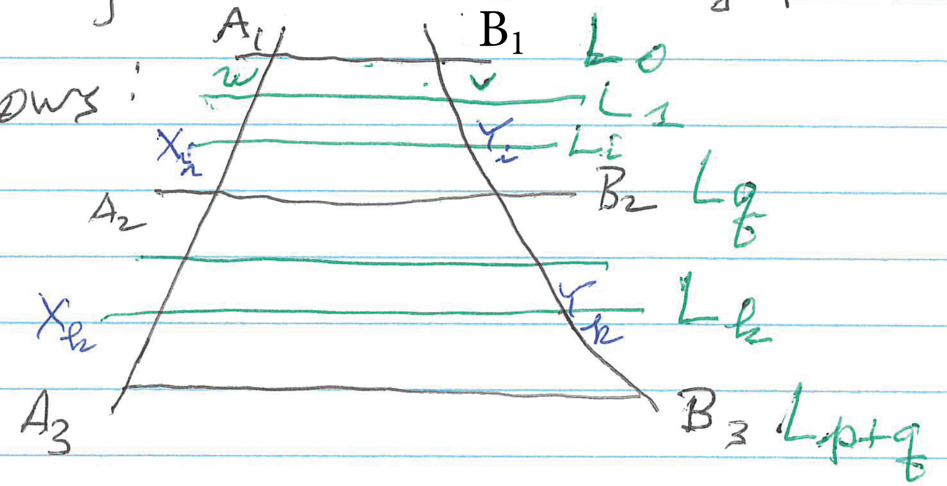


Apply preceding successively to L_{i-1}, L_i, L_{i+1}

<u>Rational Proportionality Thm.</u> L_1 L_2 L_3 as before $A_i \in M_1 \cap L_i$, $B_i \in M_2 \cap L_i$. If $\frac{|A_2A_3|}{|A_1A_2|}$ is rational, then ~~so is~~ $\frac{|B_2B_3|}{|B_1B_2|} = \frac{|A_2A_3|}{|A_1A_2|}$.

By algebra we also have $\frac{|B_1B_2|}{|A_1A_2|} = \frac{|B_2B_3|}{|A_2A_3|}$.

Proof Let $\frac{|A_2A_3|}{|A_1A_2|} = \frac{p}{q}$, so $\frac{|A_2A_3|}{p} = \frac{|A_1A_2|}{q} = w$ ("unit length"). Draw auxiliary parallels L_i as follows:



$\boxed{|X_{i+1}X_i| = w \text{ all } i}$

Previous results show that $|Y_{i+1}Y_i| = v$ all i, so also by betweenness

$$\left.\begin{array}{l} |A_1A_2| = qw, \quad |A_2A_3| = pw \\ |B_1B_2| = qv, \quad |B_2B_3| = pv, \end{array}\right\} \text{so } \frac{|B_2B_3|}{|B_1B_2|} = q.$$

Irrational $\frac{A_2A_3}{|A_1A_2|}$. Need more sophisticated ideas, took Greeks about 200 years to find a solution.

<u>Condition of Eudoxus</u> $0 < x, y \in \mathbb{R}$.
Then $x = y \iff$
(a) every positive rational $\frac{p}{q} < x$ satisfies $\frac{p}{q} < y$.
(b) Same with inequality directions reversed.
<u>Key idea</u> If $0 < x < y$ there is some $\frac{p}{q}$ so $x < \frac{p}{q} < y$. (Look at decimal expansions, for example).

$\boxed{\text{See Moise 11.3–11.4 for details}}$
read & understand passively!

<u>Similar Triangles</u> $\triangle ABC \underset{r}{\sim} \triangle DEF$ vertices ordered
(Similar, with ratio of similitude $= r$) $\iff$
$|\measuredangle ABC| = |\measuredangle DEF|$, $|\measuredangle BCA| = |\measuredangle DFE|$, $|\measuredangle CAB| = |\measuredangle FDE|$

and $$\frac{|DE|}{|AB|} = \frac{|DF|}{|AC|} = \frac{|EF|}{|BC|} = r \begin{pmatrix} \text{some} \\ r \end{pmatrix}.$$

<u>Abstract stuff</u>. $\triangle ABC \cong \triangle DEF \iff$
$\triangle ABC \underset{1}{\sim} \triangle DEF$

$\triangle ABC \underset{1}{\sim} \triangle ABC$, $\triangle ABC \underset{r}{\sim} \triangle DEF \Rightarrow \triangle DEF \underset{1/r}{\sim} \triangle ABC$

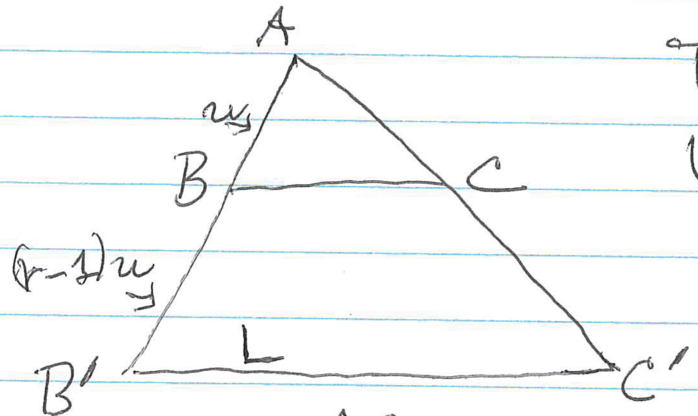
$\triangle ABC \underset{r}{\sim} \triangle DEF$ & $\triangle DEF \underset{s}{\sim} \triangle GHK \Rightarrow \triangle ABC \underset{rs}{\sim} \triangle GHK$

AA Similarity Theorem Given $\triangle ABC$ & $\triangle DEF$ s.t. $|\angle BAC| = |\angle EDF|$, $|\angle ABC| = |\angle DEF|$. Then $\triangle ABC \sim \triangle DEF$.

Proof. We also have $|\angle ACD| = |\angle DFE|$.

Let $r = \frac{|DE|}{|AB|}$. ~~If $r=1$ then $\triangle ABC \cong \triangle DEF$.~~

Now suppose $r \neq 1$. Switching the roles of the triangles if nec., can assume $r \geq 1$.



Take $B' \in \overrightarrow{AB}$ so $|AB'| = r|AB|$. Let line L through B' so that $L \parallel BC$. Notice $A*B*B'$ $(r>1)$

CLAIM L & AC not parallel; otherwise

$L \parallel AC$ & $L \parallel BC \Rightarrow BC \parallel AC$, false since these meet at C. Suppose L & AC meet at C'. Then $L \subseteq B'$-side AC & $A*B*B' \Rightarrow C'$ & A on opp sides, So C, C' on same side and $A*C*C'$. By prev. thm.

$$\frac{|CC'|}{|AC|} = \frac{|BB'|}{|AB|} = r-1, \text{ so } \frac{|AB'|}{|AB|} \underset{A*B*B'}{=} \frac{|AB|+|BB'|}{|AB|}$$

$$= \frac{|BB'|}{|AB|} + 1; \text{ likewise } \frac{|AC'|}{|AC|} = \frac{|CC'|}{|AC|} + 1. \text{ Since}$$

$$\frac{|CC'|}{|AC|} = \frac{|BB'|}{|AB|}, \text{ get } r = \frac{|AB'|}{|AB|} = \frac{|AC'|}{|AC|}.$$

Also $\measuredangle BAC = \measuredangle B'AC'$, $|\measuredangle ABC| = |\measuredangle AB'C'|$, $|\measuredangle ACB| = |\measuredangle AC'B'|$. Combining with hypotheses, get $|\measuredangle B'AC'| = |\measuredangle EDF|$, $|\measuredangle AB'C'| = |\measuredangle DEF|$, and

$|DE| = r|AB| = |AB'|$, so $\Delta AB'C' \cong \Delta DEF$ by SAS.

Will suffice to prove $\Delta ABC \sim \Delta DEF$. Have shown

AAA & $\frac{|AB'|}{|AB|} = \frac{|AC'|}{|AC|}$. If we substitute $A \to B^*$, $B \to C^*$, $C \to A^*$,

equiv to $\frac{|DE|}{|AB|} = \frac{|DF|}{|AC|}$

$D \to E'$, $E \to F'$, $F \to D'$. Then $\frac{|D^*F^*|}{|A^*C^*|} = \frac{|D^*E^*|}{|A^*B^*|}$ or $\frac{|EF|}{|BC|} = \frac{|DF|}{|AC|}$.

Hence $\Delta ABC \sim \Delta AB'C' \cong \Delta DEF$.

Construction yields

<u>Cor.</u> Given ΔABC & $r \geq 1$. Then can find $\Delta AB'C'$ so $B' \in (AB$, $C' \in (AC$, $\Delta ABC \underset{r}{\sim} \Delta AB'C'$.