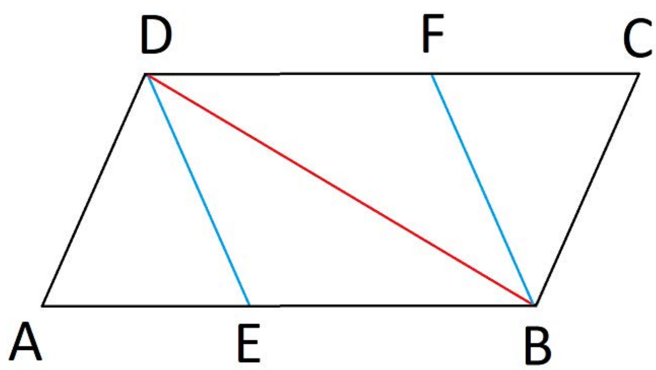
Math 133
Fall 2021

SOLUTIONS FOR WEEK 04 EXERCISES

Assume that $(\mathbf{S}; \mathcal{P}; \mathcal{L}; d; \alpha)$ or $(\mathbf{P}; \mathcal{L}; d; \alpha)$ is a system which satisfies the axioms for Euclidean geometry.

1. Here is a drawing of a typical case:



We shall first answer the question in the hint and show that $DE \neq BF$. If these lines were equal then both would be the diagonal line DB. Since this line meets each of AB and CD in at most one point, we would then have $E = A$ and $F = C$. This contadicts our assumptions that E and F lie on the closed segments (AB) and (CD), so we can conclude that $DE \neq BF$.

One step in proof of the basic measurement identities for parallelograms is proving that $\triangle ADB \cong \triangle CBD$ and $\triangle ADC \cong \triangle CBA$. This yields the identity

$$|\angle ADB| \;=\; |\angle DBC| \;.$$

The betweenness conditions $A * E * B$ and $C * F * D$ imply that E and F lie in the interiors of $\angle ADB$ and $\angle CBD$ respectively, so the additivity of angle measures imply that

$$|\angle ADE| \;+\; |\angle EDB| \;=\; |\angle ADB| \;=\; |\angle CBD| \;=\; |\angle BDF| \;+\; |\angle FBC| \;.$$

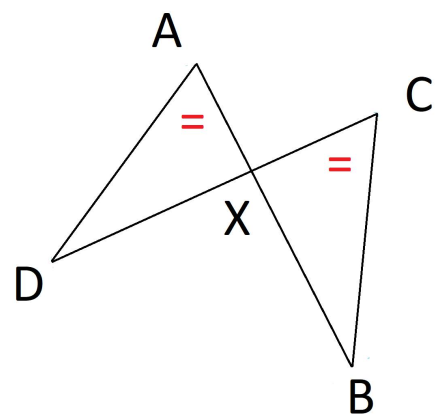
By construction and the first display in this argument we also have

$$|\angle ADE| \;=\; \frac{1}{2}\,|\angle ADB| \;=\; \frac{1}{2}\,|\angle DBC| \;=\; |\angle FBC|$$

and if we subtract this equation from the previous one we have $|\angle EDB| = |\angle FBD|$.

The drawing suggest that the two angles in the preceding sentence are alternate interior angles, and the next step is to verify this. Since the diagonals of a parallelogram have a point in common, we know that A and C lie on opposite sides of BD, and the betweenness conditions $A * E * B$ and $C * F * D$ imply that E and F lie on the same sides of BD as A and C respectively. Therefore E and F lie on opposite sides of BD, so that $\angle EDB$ and $\angle FBD$ are alternate interior angles. The latter implies that DE and BF are parallel lines.■

2. Here is a sketch:



By the Vertical Angle Theorem we know that $|\angle AXD| = |\angle CXB|$, and hence by the AA Similarity Theorem we have $\triangle AXD \sim \triangle CXB$. The latter yields the proportionality equation

$$\frac{|AX|}{|CX|} = \frac{|BX|}{|DX|}$$

and if we clear this of fractions we find that $|AX| \cdot |XB| = |CX| \cdot |XD|$.■

3. (*a*) Since $\triangle ABC \sim \triangle BCA$, let the ratio of similitude be r. If we permute the vertices on both sides cyclically and consistently, we see that $\triangle BCA \sim \triangle CAB$. Furthermore, if r' is the ratio of similitude for the second similarity then the two similarities yield

$$\frac{|AB|}{|BC|} = r \qquad \text{and} \qquad \frac{|AB|}{|AC|} = r'$$

and consequently $r = r'$. Likewise, the same cyclic permuation now yields $\triangle CAB \sim \triangle ABC$ with ratio of similitude r. By the transitivity of similarity we have $\triangle ABC \sim \triangle ABC$ with ratio of similitude r^3.

The preceding sentence yields $|AB| = r^3 \cdot |AB|$, and since $r > 0$ it follows that $r = 1$. If we substitute this back into the original similarity we find that

$$\frac{|AB|}{|BC|} = 1 \qquad \text{and} \qquad \frac{|BC|}{|AC|} = 1$$

and therefore $\triangle ABC$ is equilateral.■

(*b*) Since $\triangle ABC \sim \triangle ACB$, let the ratio of similitude be r. Then we also have $\triangle ABC \sim \triangle ACB$ with ratio of similitude equal to $1/r$, and as in the preceding argument it follows that $r = 1/r$. The only solution to this equation over the positive reals is $r = 1$, and therefore we have $\triangle ABC \cong \triangle ACB$, so that the triangle isosceles with $|AB| = |AC|$.■

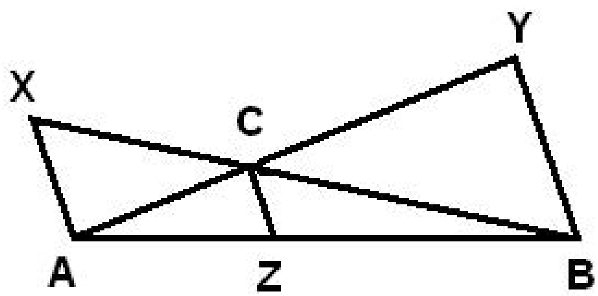
Note, In the second part the triangle need not be equilateral; one easy example is a 45–45–90 isosceles right triangle.

4. Let L and M be lines through A and D which are parallel to BC. By Pasch's Theorem we know that M must contain a point F of $[BC]$ or $[AC]$, and by construction this point cannot lie on either L or BC. Therefore the intersection point F must lie on (AC). By the Notebook Paper Theorem we know that $|AF| = |FC|$, and $A * F * C$ implies that $|AC| = 2\,|AF|$; this means that F must be the midpoint and hence $F = E$. Therefore we know that $DE = M$ and hence DE is parallel to BC. By the SAS similarity theorem we also have $\triangle ADE \sim \triangle ABC$ with ratio of similitude $\frac{1}{2}$. Since $|AD| = \frac{1}{2}\,|AB|$, we must also have $|DE| = \frac{1}{2}\,|BC|$.■

5. Suppose that r is the common ratio, so that $a_i = rb_i$ for all i. Adding these up, we find that $a_1 + ... + a_n = r(b_1 + ... + b_n)$ and hence

$$\frac{a_1}{b_1} \quad = \quad r \quad = \quad \frac{a_1 + ... + a_n}{b_1 + ... + b_n} \;.\blacksquare$$

6. Here is a sketch:



To simplify the algebraic expressions let $a = |AX|$, $b = |BY|$, $c = |CZ|$, $u = |AZ|$ and $v = |ZB|$. Since $A * Z * B$ holds, it follows that $|AB| = u + v$.

Since CZ is parallel to AX, we have $\triangle BCZ \sim \triangle BAX$, so that

$$\frac{c}{v} \quad = \quad \frac{b}{u+v}$$

and likewise since CZ is parallel to BY, we have $\triangle ACZ \sim \triangle ABY$, so that

$$\frac{c}{u} \quad = \quad \frac{a}{u+v} \;.$$

Since $p/q = r/s$ if and only if $p/r = a/x$, the preceding proportions imply

$$\frac{c}{a} \quad = \quad \frac{u}{u+v}\;, \qquad\qquad \frac{c}{b} \quad = \quad \frac{v}{u+v}$$

and if we add these equations we obtain

$$c \cdot \left(\frac{1}{a} \;+\; \frac{1}{b}\right) \quad = \quad \frac{u+v}{u+v} \quad = \quad 1$$

and if we divide both sides of this equation by c we obtain

$$\frac{1}{a} + \frac{1}{b} \quad = \quad \frac{1}{c}$$

which is equivalent to the equation in the conclusion of the exercise.■

Note. This picture is closely related to the derivation of the *thin lens formula* in elementary physics. An interactive derivation of this formula is given at the following online site:

`http://www.hirophysics.com/Anime/thinlenseq.html`