Math 133
Fall 2021

## MORE SOLUTIONS FOR WEEK 04 EXERCISES

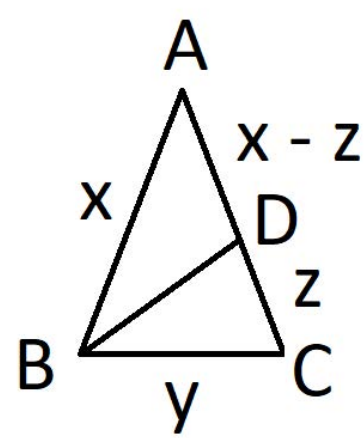
For the these exercises assume that $(\mathbf{S}; \mathcal{P}; \mathcal{L}; d; \alpha)$ or $(\mathbf{P}; \mathcal{L}; d; \alpha)$ is a system which satisfies the axioms for Euclidean geometry.

**7.** $(a)$ Let $L$ and $M$ be perpendicular lines, and let $L \cap M = \{C\}$. Then there are two points $A, A' \in L$ so that $|AC| = |A'C| = b$, and there are also two points $B, B' \in M$ so that $|BC| = |B'C| = a$. Then $\triangle ABC$ satisfies $|BC| = a$, $|AC| = b$ and $|\angle ACB| = 90°$. This result is pretty straightforward, but it is needed for $(b)$.■

$(b)$ The Hinge Theorem states that if $\triangle ABC$ and $\triangle DEF$ satisfy $|AC| = |DF|$ and $|BC| = |EF|$, then $|\angle ACB| < |\angle DFE|$ if and only if $|AB| < |DE|$. We may switch the roles of the two triangles to obtain the following companion result: If $\triangle ABC$ and $\triangle DEF$ satisfy $|AC| = |DF|$ and $|BC| = |EF|$, then $|\angle ACB| > |\angle DFE|$ if and only if $|AB| > |DE|$.

By $(a)$ we know that there is a right triangle $\triangle DEF$ such that $|AC| = |DF|$, $|BC| = |EF|$ and $|\angle DFE| = 90°$. Therefore $|DE|^2 = |DF|^2 + |EF|^2 = |AC|^2 + |BC|^2$ by the Pythagorean Theorem and the construction of $\triangle DEF$. Since two positive real numbers $u$ and $v$ satisfy $u < v$ if and only if $u^2 < v^2$, it follows that $|\angle ACB| < |\angle DFE| = 90°$ if and only if $|AB|^2 < |AC|^2 + |BC|^2$ and $|\angle ACB| > 90°|\angle DFE| = 90°$ if and only if $|AB|^2 > |AC|^2 + |BC|^2$.■

**8.** Here is a drawing:




Since $[BD$ bisects $\angle ABC$, the angle bisector theorem implies that

$$\frac{|BC|}{|AB|} = \frac{|CD|}{|AD|}$$

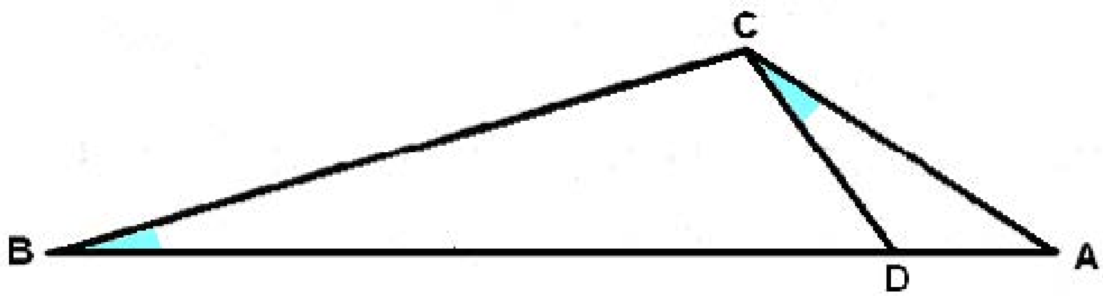
and since $A * D * C$ holds by assumption we have $|AD| = |AC| - |CD| = x - z$, so that the ratio equation becomes

$$\frac{y}{x} = \frac{z}{x-z} .$$

If we clear fractions in this equation we obtain $yx - yz = zx$, and if we solve this for $z$ we obtain

$$z = \frac{xy}{x+y} .\blacksquare$$

**9.** Here is a drawing:

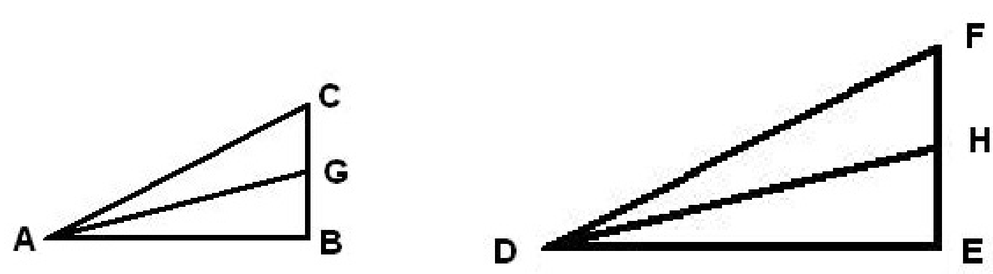



The hypothesis on angle measures and $\angle DAC = \angle CAB$ imply that $\triangle DAC \sim \triangle CAB$ by the AA Similarity Theorem. This in turn implies that

$$\frac{|AB|}{|AC|} = \frac{|AC|}{|AD|}$$

and if we multiply by both denominators we find that $|AC|^2 = |AB| \cdot |AD|$, which is what we wanted to prove.$\blacksquare$

**10.** Here is a drawing:




Let $r$ be the ratio of similitude, so that

$$\frac{|DE|}{|AB|} = \frac{|EF|}{|BC|} = \frac{|DF|)}{|AC|} = r$$

and note that $\angle GBA = \angle CBA$ and $\angle HED = \angle FED$, so that $|\angle ABG| = |\angle DEH|$. Since $G$ and $H$ are midpoints of $[BC]$ and $[EF]$, we have $|BG| = \frac{1}{2}\,|BC|$ and $|EH| = \frac{1}{2}\,|EF|$, so that

$$\frac{|EH|}{|BG|} \quad = \quad \frac{|EF|}{|BC|} \quad = \quad r \ .$$

Combining these, we can apply the SAS similarity theorem to conclude that $\triangle ABG \sim \triangle DEH$ with ratio of similitude $r$.■

**11.** We know that $\triangle ABC$ is isosceles, and by permuting the vertices if necessary we may assume that $|AB| = |AC|$ and $\triangle ABC \sim \triangle DEF$. Therefore we have

$$\frac{|AB|}{|DE|} \quad = \quad \frac{|AC|}{|DF|} \qquad \text{and thus also} \qquad \frac{|AB|}{|AC|} \quad = \quad \frac{|DE|}{|DF|} \ .$$

Since $\triangle ABC$ is isosceles the left hand side of this equation is equal to 1, which means that $|DE| = |DF|$ and hence $\triangle DEF$ is also isosceles.■

**12.** By the main result on setting up coordinates, we can construct a coordinate system for an arbitrary pair of perpendicular lines $L$ and $M$ and ruler functions $f : L \to \mathbb{R}$ and $g : M \to \mathbb{R}$; in this construction the common point of $L$ and $M$ corresponds to $(0, 0)$. Starting with noncollinear points $A$, $B$, $C$ in the given plane, take $L = AB$ and $M$ to be the perpendicular to $L$ at $A$, and let $f_0 : L \to \mathbb{R}$ and $g_0 : M \to \mathbb{R}$ be ruler functions for $L$ and $M$ respectively such that $f_0(A) = 0 = g_0(A)$.

If $D$ is the foot of the perpendicular $L'$ to $M$ through $C$, we claim that $D \notin L$. First note that $C \in L'$ but $C \notin L$ by construction. Now $L \neq L'$, and since both are perpendicular to $M$ it follows that $L'$ is parallel to $L$ hence $D \notin L$, so that $D \neq A$. Therefore $D \notin L$ as claimed. Multiplying each of $f_0$ and $g_0$ by $-1$ if necessary, we obtain ruler functions $f$ and $g$ such that $f(B) > 0$ and $g(C) = g(D) > 0$. It follows that $A$ corresponds to $(0, 0)$, $B$ corresponds to $(u, 0)$ for some $u > 0$, and $C$ corresponds to $(x, y)$ where $y > 0$.■