Math 133
Fall 2021

EXERCISES FOR WEEK 11

For these exercises assume that the coordinate plane satisfies the axioms for Euclidean plane geometry, with distances and lines as described in Chapter 17 of Moise. We shall not need the angular measure concept or its consequences explicitly.

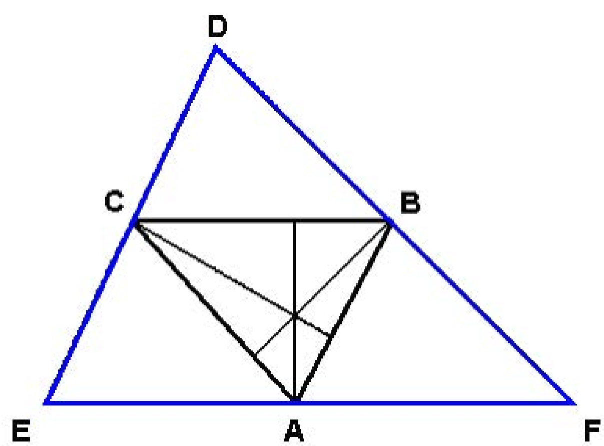
0. Suppose that $\mathbf{a} = (a_1, a_2)$ and $\mathbf{b} = (b_1, b_2)$ are two points in the coordinate plane in vector form such that $a_1 \neq b_1$, and let $y = mx + c$ be an equation defining L. Verify that L is defined in vector form by the set of all vectors $\mathbf{x}$ such that $\mathbf{x} = \mathbf{a} + t(\mathbf{b} - \mathbf{a})$ for some real number t. [*Hint:* One can solve for m and c in terms of the coordinates of $\mathbf{a}$ and $\mathbf{b}$ in Moise or precalculus courses.]

1. Let L be the line in the coordinate plane which is defined by $y = mx + b$ where m is some real number. Prove that the function f from L to the real line defined by

$$f(x, y) \quad = \quad x \cdot \sqrt{1 + m^2}$$

is a ruler function.

2. Prove that the three altitudes of a triangle have a point in common (the *orthocenter*). — The standard method is to start with a triangle $\triangle ABC$ and construct a new triangle $\triangle DEF$ as illustrated below. More precisely, A, B and C are the midpoints of EF, DF, and DE respectively, and EF, DF, and DE are parallel to BC, AC, and AB respectively. One important step is to prove that such a triangle exists. In vector terms, one approach is to take $D = B + C - A$, $E = A + C - B$, and $F = A + B - C$.



3. (a) Let L be a line in the coordinate plane, and suppose that we have two decompositions $\{H_1, H_2\}$ and $\{H_1^*, H_2^*\}$ for the complement of L which satisfy the conditions of the plane separation postulate. Prove that $\{H_1, H_2\} = \{H_1^*, H_2^*\}$.

(b) Assume that the line L in the coordinate plane is defined by an equation of the form $y = mx + b$. Prove that the sets

$$\begin{aligned} H_+ &= \{(x,y) \mid y > mx + b\} \\ H_- &= \{(x,y) \mid y < mx + b\} \end{aligned}$$

satisfy the conditions in the Plane Separation Postulate.

4. (a) Suppose that we are given four points A, B, C, D in the coordinate plane such that no three are collinear, and assume that X is the midpoint of both $[AC]$ and $[BD]$. Prove that A, B, C, D (in that order) are the vertices of a parallelogram.

(b) Suppose that we are given four points A, B, C, D in the coordinate plane such that no three are collinear and $C - D = B - A$. Prove that A, B, C, D (in that order) are the vertices of a parallelogram.

5. Suppose that we are given points in the coordinate plane $U = (a_1, b_1)$ and $V = (a_2, b_2)$ with $A \neq B$. Verify that the line joining A to B is the set of all (x, y) such that

$$\begin{vmatrix} a_1 & a_2 & x \\ b_1 & b_2 & y \\ 1 & 1 & 1 \end{vmatrix} = 0$$

where the object on the left is a 3×3 determinant.

Reminder: To show that an equation of the form $Px + Qy + R = 0$ defines a line, you need to check that at least one of the coefficients P and Q is nonzero.

6. Prove the **Theorem of Menelaus:** *Let* $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ *be the vector forms of three noncollinear points, and suppose we are given points* $\mathbf{p} \in \mathbf{bc}$, $\mathbf{q} \in \mathbf{ac}$, *and* $\mathbf{r} \in \mathbf{ab}$ *which are not equal to* $\mathbf{c}$, $\mathbf{a}$, *and* $\mathbf{b}$ *respectively. Write the vectors* $\mathbf{p}$, $\mathbf{q}$, $\mathbf{r}$ *as*

$$\mathbf{p} = \mathbf{b} + t(\mathbf{c} - \mathbf{b}) = t\mathbf{c} + (1-t)\mathbf{b}$$

$$\mathbf{q} = \mathbf{c} + u(\mathbf{a} - \mathbf{c}) = u\mathbf{a} + (1-u)\mathbf{c}$$

$$\mathbf{r} = \mathbf{a} + v(\mathbf{b} - \mathbf{a}) = v\mathbf{b} + (1-v)\mathbf{a}$$

where t, u, *and* v *are appropriate scalars, none of which is equal to* 0 *or* 1. *Then* $\mathbf{p}$, $\mathbf{q}$, *and* $\mathbf{r}$ *are collinear if and only if*

$$\frac{tuv}{(1-t)(1-u)(1-v)} = -1 \; .$$

[*Hint:* Use the preceding exercise.]