Using coordinates II – vectors

Vectors are a powerful tool for working with coordinates.

EXAMPLE Midpoint M of [AB].

$|AM| = |MB| = \frac{1}{2}|AB|$ in coords. yields

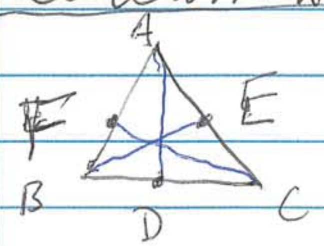
$(m_1, m_2, ?m_3) = (\frac{1}{2}(a_1+b_1), \frac{1}{2}(a_2+b_2), ?\frac{1}{2}(a_3+b_3))$

2D & 3D slightly different.

Merge using vectors m, a, b with coords m_1, m_2, etc. $m = \frac{1}{2}(a+b)$.

Check $|b-m| = |\frac{1}{2}(b-a)| = |a-m|$.

Concurrence of Δ medians



AD, BE, CF meet at point G so

$|AG| = \frac{2}{3}|AD|$, $|BG| = \frac{2}{3}|BE|$

$|CG| = \frac{2}{3}|CF|$.

Proof Choose coords so $A \leftrightarrow (0,\alpha)$

$A \leftrightarrow \alpha$ $\quad D = \frac{1}{2}(\beta+\gamma)$

$B \leftrightarrow \beta \quad \Rightarrow \quad E = \frac{1}{2}(\alpha+\gamma)$

$C \leftrightarrow \gamma$ $\quad F = \frac{1}{2}(\alpha+\beta)$.

Neither B nor C is a mult of other

AD meets BE at G =

$\alpha + p\left(\frac{1}{2}\beta + \frac{1}{2}\gamma - \alpha\right) = \beta + q\left(\frac{1}{2}\alpha + \frac{1}{2}\gamma - \beta\right)$

$\frac{p}{2}\beta + \frac{p}{2}\gamma = \frac{q}{2}\gamma + (1-q)\beta$.

equating coeffs. $p = q$ and $1 - q = \frac{q}{2}$

$2 - 2q = q$
$2 = 3q$
$q = \frac{2}{3}$.

So $|AG| = \frac{2}{3}|AB|$ & $|BG| = \frac{2}{3}|BE|$ and

$G = \frac{1}{3}(\alpha+\beta+\gamma)$

Switching the roles of B + C, see that

AD meets CF also at $\frac{1}{3}(\alpha+\beta+\gamma)$.

Existence of circumscribed circle for ΔABC

Want X so $|X-A| = |X-B| = |X-C|$.

Equivalently, can square these.

$$|X|^2 - 2A\cdot X + |A|^2 = |X|^2 - 2B\cdot X - |B|^2$$
$$|X|^2 - 2A\cdot X + |A|^2 = |X|^2 - 2C\cdot X + |C|^2,$$

Choose coords so $A \leftrightarrow 0$. Get two eqns.

a, x, b, c vectors

$$|b|^2 - 2b\cdot x = 0 \quad \Big| \quad 2b\cdot x = |b|^2$$
$$|c|^2 - 2c\cdot x = 0 \quad \Big| \quad 2c\cdot x = |c|^2$$

b + c not multiples of each other, so there is a unique solution for x in the plane.

<u>"Aristotle's Theorem."</u> [in his work on meteorology — the science is wrong but the proof is mathematically correct]

$A \neq B$ points, $r > 0$ but $r \neq 1$. Then the set of all points X so that $|BX| = r\,|AX|$ is a circle. (Suffices to do case $r < 1$; switch roles of A + B if $r > 1$)

<u>Proof</u> Choose coords so $A \leftrightarrow 0$ and then $|BX| = r|X| \Leftrightarrow |B - X|^2 = r^2|X|^2$

and hence $|B|^2 - 2B\cdot X + |X|^2 = r^2|X|^2$.

Let $B \leftrightarrow (b, 0)$ where $b > 0$.

If $X=(x,y)$, then eqn becomes

$$(1-r^2)|X|^2 - 2bx + b^2 = 0$$

$$x^2+y^2 - \frac{2b}{1-r^2}x + \frac{b^2}{1-r^2} = 0$$

Complete square on left:

$$\left(x-\frac{b}{1-r^2}\right)^2 + y^2 = \frac{-b^2}{1-r^2} + \frac{b^2}{(1-r^2)^2}$$

Now if $r<1$ & $r>0$ then

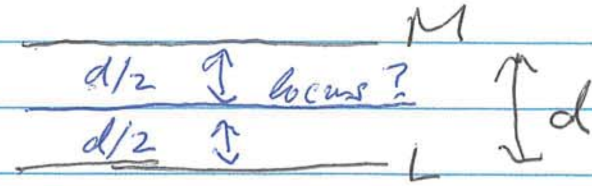
$(1-r^2)^2 < 1-r^2$ so $\frac{1}{(1-r^2)^2} > \frac{1}{1-r^2}$

so the right side of the eqn is positive and the eqn defines a circle.

(compare with case $r=1$; get a line).

Another locus problem Given $\parallel$ lines L & M and pos real number r, what is the locus (= set) of all X so distance $(X,L) = r\,d(X,M)$?

Say $r = 1$ first.



$X = (u, v) \in \mathbb{R}^2$

$d(X, L) = |u| \quad d(Y, L) = |d - u|$

So equation is $|u| = |d - u|$ or

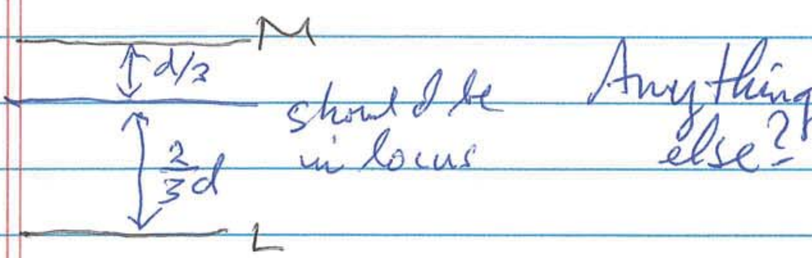
$u^2 = (d-u)^2$. Subtract u^2 from both sides

Get $0 = d^2 - 2du$, or $2u =$ d

Hence $u = \frac{d}{2}$. Retrace to get converse

What if $r \neq 1$?

$r = 2$



The defining eqn. for this set is

$$|u| = 2|d-u|$$ equivalently

$$u^2 = 4(u^2 - 2du + d^2)$$ or

$$0 = 3u^2 - 8du + 4d^2$$

Apply the quadratic formula

$$u = \frac{8d \pm \sqrt{64d^2 - 48d^2}}{6} = \frac{8 \pm 4}{6} d$$

So $u = \frac{2}{3}d$ or $2d$ and there are two parallel lines in the set.

More generally, suppose $r > 1$. Then

$$u^2 = r^2(u^2 - 2ud + d^2)$$

$$0 = (r^2-1)u^2 - 2r^2du + r^2d^2$$

$$u = \frac{2r^2d \pm \sqrt{4r^4d^2 - 4(r^2-1)r^2d^2}}{2(r^2-1)} =$$

$$\frac{r^2d \pm d\sqrt{r^4 - r^4 + r^2}}{r^2-1} = \frac{r^2 \pm r}{r^2-1} d$$

So $u = \frac{rd}{r+1}$ or $u = \frac{rd}{r-1}$.