Vectors and Parallel Lines

Theorem Let AB and CD be lines in $\mathbb{R}^2$. Then the following are equivalent:

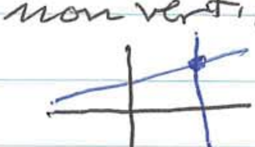
(i) Either $AB \parallel CD$ or $AB = CD$

(ii) $B-A$ and $D-C$ are nonzero multiples of each other.

This approach relies heavily on Moise, Chapter 17. Given a point X, write $X = (x_1, x_2)$.

Derivation. There are two cases: Lines vertical and nonvertical. We need

Claim: A vertical and a nonvertical line meet at one point.



Verification Let $x = a$ be the vertical line $y = mx + b$ the nonvertical one. Then the only common pt. is $(a, ma + b)$.

VERTICAL CASE (i) $\Rightarrow$ (ii) Lines are $x = a$ & $x = b$. So we have

$A = (a, y_1)$, $B = (a, y_2)$, $C = (b, y_3)$, $D = (c, y_4)$ where $y_1 \neq y_2$ + $y_3 \neq y_4$. Both $B-A$ and $D-C$ are nonzero multiples of $(0,1)$.

(ii) ⇒ (i) Two vertical lines are always equal or parallel. ■

NONVERTICAL CASE. Say the lines have equations

$$y = mx + p$$
$$y = nx + q$$

By Moise they are parallel or equal ⟺ $m = n$. Also two points on such a line are equal ⟺ first coords equal because there is a formula for finding the second coord. from the first.

(i) ⇒ (ii) Given $m = n$. Then

$$A = (a_1, ma_1 + p)$$
$$B = (b_1, mb_1 + p)$$
$$C = (c_1, mc_1 + q)$$
$$D = (d_1, md_1 + q)$$

So

$$B - A = (b_1 - a_1, m(b_1 - a_1))$$
$$D - C = (d_1 - c_1, m(d_1 - c_1))$$

Now $b_1 \neq a_1$ since $B \neq A$ and $d_1 \neq c_1$ since $D = A$. Therefore

$$D - C = \frac{d_1 - c_1}{b_1 - a_1}(B - A)$$ and

$$B - A = \frac{b_1 - a_1}{d_1 - c_1}(D - C)$$

relies heavily on fact that fraction denominators are nonzero!

(ii) $\Rightarrow$ (i) Say $D - C = k(B - A)$ where $k \neq 0$. Then

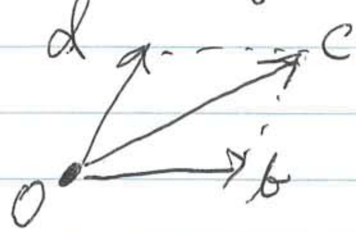
$$d_1 - c_1 = k(b_1 - a_1)$$

$$k(b_2 - a_2) = d_2 - c_2 = n(d_1 - c_1)$$ But also

$$b_2 - a_2 = m(b_1 - a_1),$$ so

$km(b_1 - a_1) = k(b_2 - a_2) = n(d_1 - c_1) = kn(b_1 - a_1)$. Since $b_1 - a_1 \neq 0$ and $k = 0$, we must have $m = n$. Hence $AB \parallel CD$ or $AB = CD$. $\blacksquare$

Parallelogram Law for Vector Addition



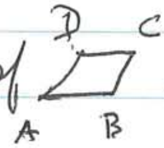
If $0, b, c, d$ (in order) are the vertices of a parallelogram, then $c = b + d$.

Derivation Show $0b \parallel cd$
$0d \parallel bc$.

Assuming b + d aren't nonzero scalar multiples of each other. Then $cd \neq 0b$ and $bc \neq 0d$. ($d \notin 0b$, $b \notin 0d$).

$d - c = -b$
apply thm to get
$cd \parallel 0b$

$b - c = -d$
apply thm to get
$bc \parallel 0d$.

Example. The diagonals of 

bisect each other

Proof. Previous yields

so $C + A = (D-A) + (B-A) + 2A$

$(C - A) = (D-A) + (B-A)$. Midpoints are

AC: $\frac{1}{2}(C+A) = \frac{1}{2}(D-A) + \frac{1}{2}(B-A) + \frac{1}{2}(2A) =$

$\frac{1}{2}(D+B)$

BD: $\frac{1}{2}(D+B)$. So we get the same point!