Math 133
Fall 2021

SOLUTIONS FOR WEEK 06 EXERCISES

For these exercises assume that the coordinate plane satisfies the axioms for Euclidean plane geometry, with distances and lines as described in Chapter 17 of Moise. We shall not need the angular measure concept or its consequences explicitly.

0. The standard rules of coordinate geometry give the following identities for the equation of the line $y = mx + c$ joining $\mathbf{a} = (a_1, a_2)$ and $\mathbf{b} = (b_1, b_2)$:

$$\frac{b_2 - a_2}{b_1 - a_1} \quad = \quad m \ , \quad y - a_2 \quad = \quad m(x - a_1) \ , \quad c \quad = \quad a_2 - ma_1$$

We shall first verify that every vector of the form $\mathbf{a} + t(\mathbf{b} - \mathbf{a})$ corresponds to a point of the line $\mathbf{ab}$. This follows because the equations

$$a_2 \quad = \quad ma_1 \ + \ c \ , \qquad b_2 \quad = \quad mb_1 \ + \ c$$

(valid since the two points lie on the line with equation $y = mx + c$) and elementary algebra imply that

$$a_2 \ + \ t(b_2 - a_2) \quad = \quad m\,(a_1 \ + \ t(b_1 - a_1)) \ + \ c \ .$$

Conversely, we claim that every point (u, v) on the line $y = mx + c$ is expressible as $\mathbf{a} + t(\mathbf{b} - \mathbf{a})$ for some real number t. The main task is to find t in terms of u and the data from the original vectors:

$$u \quad = \quad a_1 \ + \ t(b_1 - a_1)$$

It follows that

$$t \quad = \quad \frac{u - a_1}{b_1 - a_1}$$

and since $v = mu + c$ one can verify directly that $v = m\,(a_2 \ + \ t(b_2 - a_2)) \ + \ c$.■

1. The distance between the two points $(x_1, y_1) = (x_1, mx_1 + c)$ and $(x_2, y_2) = (x_2, mx_2 + c)$ on the given line is equal to

$$\sqrt{(x_2 - x_1)^2 + m^2(x_2 - x_1)^2} \quad = \quad |x_2 - x_1| \cdot \sqrt{1 + m^2}$$

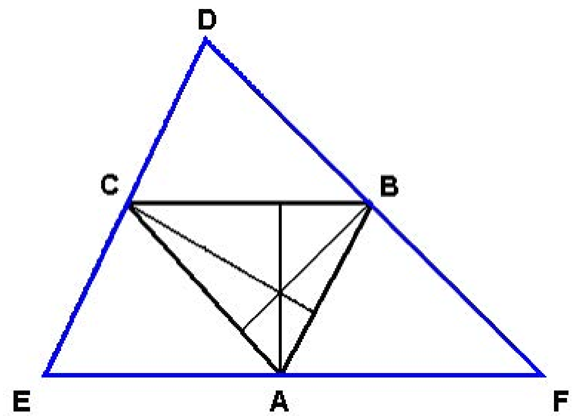
so if $A_i = (x_i, y_i)$ then $|A_1A_2| = |f(a_1) - f(A_2)|$. It remains to check that f is 1–1 and onto.

If $f(x_1, y_1) = f(x_2, y_2)$ then $x_1 \cdot \sqrt{1 + m^2} = x_2 \cdot \sqrt{1 + m^2}$, so that $x_1 = x_2$. The latter implies that $y_1 = mx_1 + c = mx_2 + c = y_2$; therefore $(x_1, y_1) = (x_2, y_2)$ and f is 1–1. Also, if r is a real number and

$$u \quad = \quad \frac{r}{\sqrt{1 + m^2}}$$

then $f(u, mu + c) = r$ and hence f is also onto.■

2. We shall follow the hint and use the drawing to guide the discussion.



We first need to verify that A, B and C are the midpoints of $[EF]$, $[DF]$ and $[DE]$ respectively. To do this, it is only necessary to expand the three vectors $\frac{1}{2}(D+F)$, $\frac{1}{2}(E+F)$, and $\frac{1}{2}(D+E)$ using the definitions above. Next, we need to show that AB is parallel to DE, AC is parallel to DF and BC is parallel to EF. It will suffice to show the following:

1. The lines AB and DE are distinct, the lines AC and DF are distinct, and the lines BC and EF are distinct.

2. The difference vectors $E-D$ and $B-A$ are nonzero multiples of each other, the difference vectors $F-D$ and $C-A$ are nonzero multiples of each other, and the difference vectors $F-E$ and $C-B$ are nonzero multiples of each other.

We can dispose of the first item as follows: Since C lies on DE and not on AB, it follows that DE and AB are distinct lines; similarly, since B lies on DF and not on AC, it follows that DF and AC are distinct lines, and finally since A lies on EF and not on BC, it follows that EF and BC are distinct lines. The assertions in the second item may be checked by expanding $E-D$, $F-D$, and $F-E$ in terms of A, B and C using the definitions. These computations yield the equations $E-D = 2(B-A)$, $F-D = 2(C-A)$, and $F-E = 2(C-B)$.

Finally, we need to verify that the altitudes M_A, M_B and M_C of $\triangle ABC$ are perpendicular to EF, DF and DE respectively. This will imply that the three lines are also the perpendicular bisectors for the sides of $\triangle DEF$, which means that the lines M_A, M_B and M_C have a point in common by a theorem from the lectures. The first perpendicularity statement follows because $M_A \perp BC$ and BC parallel to EF imply $M_A \perp EF$, the second follows because $M_B \perp AC$ and AC parallel to DF imply $M_B \perp DF$, and the third follows because $M_C \perp AB$ and AB parallel to DE imply $M_C \perp DE$.■

3. (a) It suffices to show that *either* $H_1 \subset H_1^*$ and $H_2 \subset H_2^*$ *or else* $H_1 \subset H_2^*$ and $H_2 \subset H_1^*$. In fact, it suffices to consider the first case, for the second will then follow by switching the roles of H_1^* and H_2^*. This is true because we know that $P - L$ is a union of the disjoint subsets H_1 and H_2, and it is also the union of the disjoint subsets H_1^* and H_2^*, for we can switch the roles of the primed and unprimed variables to conclude $H_1 \supset H_2^*$ and $H_2 \supset H_1^*$.

Again interchanging roles of the variables, we need only show that $H_1 \subset H_1^*$ or $H_1 \subset H_2^*$. Let $p \in H_1$; then either $p \in H_1^*$ or $p \in H_2^*$. Once again reversing the roles of variables if necessary, we reduce to considering the case where the first alternative holds.

Since no points of L are in any of the sets $\{H_1, H_2, H_1^*, H_2^*\}$, we must have

$$H_1 \;=\; (H_1 \cap H_1^*) \;\cup\; (H_1 \cap H_2^*)$$

so it suffices to show that the second summand on the right is empty. Suppose it is not, and let q be a point in this intersection. If we apply the plane separation postulate to $\{H_1^*, H_2^*\}$ we then find that there is a point $z \in L$ such that $p * z * q$. Since $p, q \in H_1$ and the latter is convex, it also follows that $z \in H_1$; this is a contradiction because the sets L and H_1 are disjoint by hypothesis. The source of the contradiction was our supposition that $H_1 \cap H_2^*$ was nonempty, so the latter is false and the intersection must indeed be empty.■

(b) We shall reformulate the problem to include the case of vertical lines: If L is a line in the coordinate plane, then L is defined by an equation of the form

$$0 \;=\; g(x,y) \;=\; Ax \;+\; By \;+\; C$$

where at least on of A, B is nonzero. Theh we want to prove that the two half-planes determined by L are the sets where $g(x,y) > 0$ and $g(x,y) < 0$; we shall denote these sets by H_1 and H_2 respectively. By (a) we need only show that these sets satisfy the conditions of the plane separation postulate (nonempty, disjoint, union is the complement of L, convex, and a line segment joining a point in one subset to a point in the other must pass through the original line).

The first thing to notice is that H_1 and H_2 are both nonempty. For each scalar k, consider the point $V_k = (kA, kB)$. We then have $g(x,y) = k(A^2 + B^2) + C$, and since at least one of A, B is nonzero it follows that the coefficient $A^2 + B^2$ is positive. Therefore we can say that $g(V_k) = g(kA, kB)$ will be positive if $k > -C/(A^2 + B^2)$ and $g(V_k) = g(kA, kB)$ will be negative if $k < -C/(A^2 + B^2)$. Since there are infinitely values of k satisfying either of these inequalities, it follows that in fact both H_1 and H_2 contain infinitely many points.

We also need to check that H_1 and H_2 are both convex; in other words, if $P = (x,y)$ and $Q = (u,v)$ belong to one of these half-planes and $0 < t < 1$, then the point $P + t(Q - P)$ also belongs to the same half-plane. The key to this is the following chain of identities:

$$g\big(\,P + t(Q-P)\,\big) \;=\; g\big(\,x + t(u-x), y + t(v-y)\,\big) \;=\; A\big(\,x + t(u-x)\,\big) \;+\; B\big(\,y + t(v-y)\,\big) \;=$$

$$(1-t)(Ax + By) \;+\; t(Ax + By) \;+\; C \;=\; (1-t)\cdot g(P) \;+\; t \cdot g(Q)\;.$$

If P and Q lie on the same side of L, then either $g(P)$ and $g(Q)$ are both positive or they are both negative. Note that t and $1-t$ are both positive in either case. If $g(P)$ and $g(Q)$ are positive, then it follows that

$$g\big(\,P + t(Q-P)\,\big) \;=\; (1-t)\cdot g(P) \;+\; t \cdot g(Q)$$

must also be positive since it is a sum of two products of positive numbers, while if $g(P)$ and $g(Q)$ are negative, then it follows that the expression is a sum of two products, each with one positive and one negative factor, and hence in this case $g\big(\,P + t(Q-P)\,\big)$ must be negative.

Finally, we need to show if P is in one half-plane and Q is in the other, then the open segment (PQ) and the line L have a point in common. In the terms of the preceding discussions, this means that we can find some t such that $0 < t < 1$ and $g\big(\,P + t(Q-P)\,\big) = 0$.

We shall only consider the case where $g(P) < 0 < g(Q)$; the other case, in which $g(P) > 0 > g(Q)$, can be obtained by interchanging the roles of P and Q in the argument below. By the fundamental identity displayed above, we need to find a value of t such that

$$0 \;=\; (1-t)g(P) \;+\; tg(Q) \;=\; g(P) \;+\; t\big(\,g(Q) - g(P)\,\big)\;.$$

The solution to this equation is

$$t = \frac{-g(P)}{g(Q) - g(P)}$$

where the denominator is positive since $g(Q) > g(P)$. By assumption $g(P)$ is negative, and therefore the entire expression for t is positive. Furthermore, we also have $0 < -g(P) < g(Q) - g(P)$, so it also follows that $t < 1$. Therefore, if we take t as given above, then the point $P + t(Q - P)$ will lie on both the open segment (PQ) and the line L.■

4. (a) By the Parallelogram Law, we need to show that $C = B + D - A$. The bisection hypotheses states that $\frac{1}{2}(A + C) = \frac{1}{2}(B + D)$, and if we multiply both sides of this equation by 2 we obtain $A + C = B + D$; if we subtract A from both sides of the latter equation we obtain the desired identity $C = B + D - A$.■

(b) If we add D to both sides of the given equation, we obtain $C = B + D - A$, so by the Parallelogram Law the points A, B, C, D (in that order) are the vertices of a parallelogram.■

The next two exercises involve 3×3 determinants. Here is a link to a Power Point file on this topic; most of what we need is on the first 19 pages.

`https://www.slideshare.net/SeyidKadher1/determinants-68070113`

We also need two additional identities, which can be verified by direct computation. It will be convenient to write a 3×3 determinant as a function of its columns. Specifically, if the three columns are given in order by E, F, G, then the determinant of the matrix will be denoted by $\mathbf{D}(E, F, G)$.

(1) If two of the columns are equal, then $\mathbf{D}(E, F, G) = 0$.

(2) We have $\mathbf{D}(E, F, G) = \mathbf{D}(F, G, E) = \mathbf{D}(G, E, F)$.

5. If we expand the determinant by minors along the third column, we obtain the following:

$$\begin{vmatrix} a_1 & a_2 & x \\ b_1 & b_2 & y \\ 1 & 1 & 1 \end{vmatrix} = x(b_1 - b_2) - y(a_1 - a_2) + (a_1 b_2 - a_2 b_1)$$

This is a nontrivial first degree equation in x and y because $(a_1, b_1) \neq (a_2, b_2)$ implies that at least one of $b_1 - b_2$ and $a_1 - a_2$ is nonzero. Direct calculation shows that the expression on the right hand side is zero if either $(x, y) = (a_1, b_1)$ or $(x, y) = (a_2, b_2)$, so the equation obtained by setting either expression equal to zero defines the line joining the original two points.■

6. Before proceeding, we shall state a reformulation of the preceding exercise: Let $X = (x_1, x_2)$, $Y = (y_1, y_2)$, and $Z = (z_1, z_2)$ be three points in the coordinate plane. Then X, Y, Z are noncollinear if and only if the determinant

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ 1 & 1 & 1 \end{vmatrix}$$

is nonzero.

Given an ordered pair $\mathbf{w} = (w_1, w_2)$ of real numbers, let $\overline{W}$ denote the 3-dimensional column vector ($= 3 \times 1$ matrix) whose entries in order are w_1, w_2 and 1. We can then reformulate the Theorem of Menelaus to say that $\mathbf{p} \in \mathbf{bc}$, $\mathbf{q} \in \mathbf{ac}$, and $\mathbf{r} \in \mathbf{ab}$ are collinear if and only if $\mathbf{D}(\overline{P}, \overline{Q}, \overline{R}) = 0$. We shall study this determinant using the following direct consequences of the definitions for $\mathbf{p}$, $\mathbf{q}$, and $\mathbf{r}$:

$$\begin{aligned}
\overline{P} &= \overline{B} + t(\overline{C} - \overline{B}) = t\overline{C} + (1-t)\overline{B} \\
\overline{Q} &= \overline{C} + u(\overline{A} - \overline{C}) = u\overline{A} + (1-u)\overline{C} \\
\overline{R} &= \overline{A} + v(\overline{B} - \overline{A}) = v\overline{B} + (1-v)\overline{A}
\end{aligned}$$

If we substitute the right hand expressions for the left hand expressions in $\mathbf{D}(\overline{P}, \overline{Q}, \overline{R})$ and apply the identities in the website and in (1) and (2) above to simplify terms, we see that the three points of interest are collinear if and only if

$$\mathbf{D}(\overline{P}, \overline{Q}, \overline{R}) \quad = \quad tuv\,\mathbf{D}(\overline{C}, \overline{A}, \overline{B}) \; + \; (1-t)(1-u)(1-v)\,\mathbf{D}(\overline{B}, \overline{C}, \overline{A}) \quad =$$

$$(\,tuv + (1-t)(1-u)(1-v)\,) \cdot \mathbf{D}(\overline{A}, \overline{B}, \overline{C}) \quad = \quad 0\ .$$

The noncollinearity of $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ implies that $\mathbf{D}(\overline{A}, \overline{B}, \overline{C}) \neq 0$, and therefore the condition for $\mathbf{p}$, $\mathbf{q}$, and $\mathbf{r}$ to be collinear reduces to the equation $(1-t)(1-u)(1-v) = -tuv$.$\blacksquare$